STAT253/317 Lecture 7

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•	Using the Recursive Relations of Markov Chains
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4.7	Branching Processes

Using the Recursive Relations of Markov Chains

Consecutive terms in many Markov chains $\{X_n\}$ often have some recursive relations like

$$X_{n+1} = g(X_n, \xi_{n+1}) \quad \text{for all } n$$

where $\{\xi_n, n = 0, 1, 2, \dots\}$ are some i.i.d. random variables and X_n is independent of $\{\xi_k : k > n\}$.

In many cases, we can use the recursive relationship to find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$ without knowing the distribution of X_n.

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n])\end{aligned}$$

Example 1: Simple Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q = 1 - p \end{cases}$$

So

$$\begin{aligned} \mathbb{E}[X_{n+1}|X_n] &= p(X_n + 1) + q(X_n - 1) = X_n + p - q \\ \mathrm{Var}[X_{n+1}|X_n] &= 4pq \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = \mathbb{E}[X_n] + p - q \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) \\ &= \mathbb{E}[4pq] + \mathrm{Var}(X_n + p - q) = 4pq + \mathrm{Var}(X_n) \end{aligned}$$

So

$$\mathbb{E}[X_n] = n(p - q) + \mathbb{E}[X_0], \qquad \mathrm{Var}(X_n) = 4npq + \mathrm{Var}(X_0)$$

Example 2: Ehrenfest Urn Model with M Balls

Recall that

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M - X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}$$

We have

$$\mathbb{E}[X_{n+1}|X_n] = (X_n+1) \times \frac{M - X_n}{M} + (X_n - 1) \times \frac{X_n}{M} = 1 + \left(1 - \frac{2}{M}\right) X_n.$$

Thus

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = 1 + \left(1 - \frac{2}{M}\right) \mathbb{E}[X_n]$$

Subtracting $M/2$ from both sides of the equation above, we get

$$\mathbb{E}[X_{n+1}] - \frac{M}{2} = \left(1 - \frac{2}{M}\right) (\mathbb{E}[X_n] - \frac{M}{2})$$

Thus

$$\mathbb{E}[X_n] - \frac{M}{2} = \left(1 - \frac{2}{M}\right)^n (\mathbb{E}[X_0] - \frac{M}{2})$$

Variance of Ehrenfest Urn Model

$$\mathbb{E}[X_{n+1}|X_n] = 1 + \left(1 - \frac{2}{M}\right) X_n, \quad \mathrm{Var}(X_{n+1}|X_n) = \frac{4X_n(M - X_n)}{M^2}$$

and hence

$$\begin{aligned}
\mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) &= \mathrm{Var}(1 + \left(1 - \frac{2}{M}\right) X_n) = \left(1 - \frac{2}{M}\right)^2 \mathrm{Var}(X_n) \\
\mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] &= \frac{4\mathbb{E}[X_n(M - X_n)]}{M^2} = \frac{4}{M}\mathbb{E}[X_n] - \frac{4}{M^2}\mathbb{E}[X_n^2] \\
&= \frac{4}{M}\mathbb{E}[X_n] - \frac{4}{M^2}\left(\mathrm{Var}(X_n) + (\mathbb{E}[X_n])^2\right) \\
&= -\frac{4}{M^2}\mathrm{Var}(X_n) + 4\frac{\mathbb{E}[X_n]}{M}\left(1 - \frac{\mathbb{E}[X_n]}{M}\right)
\end{aligned}$$

So

$$\begin{aligned}\operatorname{Var}(X_{n+1}) &= \operatorname{Var}(\mathbb{E}[X_{n+1}|X_n]) + \mathbb{E}[\operatorname{Var}(X_{n+1}|X_n)] \\ &= \left(1-\frac{2}{M}\right)^2 \operatorname{Var}(X_n) - \frac{4}{M^2}\operatorname{Var}(X_n) + 4\frac{\mathbb{E}[X_n]}{M}\left(1-\frac{\mathbb{E}[X_n]}{M}\right) \\ &= \left(1-\frac{4}{M}\right)\operatorname{Var}(X_n) + 4\frac{\mathbb{E}[X_n]}{M}\left(1-\frac{\mathbb{E}[X_n]}{M}\right)\end{aligned}$$

Recall $\mathbb{E}[X_n] = \frac{M}{2} + \left(1-\frac{2}{M}\right)^n(\mathbb{E}[X_0]-\frac{M}{2})$. So

$$\begin{aligned}\frac{\mathbb{E}[X_n]}{M} &= \frac{1}{2} + \left(1-\frac{2}{M}\right)^n\left(\frac{\mathbb{E}[X_0]}{M}-\frac{1}{2}\right), \\ 1-\frac{\mathbb{E}[X_n]}{M} &= \frac{1}{2} - \left(1-\frac{2}{M}\right)^n\left(\frac{\mathbb{E}[X_0]}{M}-\frac{1}{2}\right)\end{aligned}$$

and their product is

$$\frac{\mathbb{E}[X_n]}{M}\left(1-\frac{\mathbb{E}[X_n]}{M}\right) = \frac{1}{4} - \left(1-\frac{2}{M}\right)^{2n}\left(\frac{\mathbb{E}[X_0]}{M}-\frac{1}{2}\right)^2$$

$$\mathrm{Var}(X_{n+1}) = \left(1 - \frac{4}{M}\right)\mathrm{Var}(X_n) + 1 - \left(1 - \frac{2}{M}\right)^{2n}\left(\frac{2\mathbb{E}[X_0]}{M} - 1\right)^2$$

Subtracting $M/4$ from both sides, we get

$$\mathrm{Var}(X_{n+1}) - \frac{M}{4} = \left(1 - \frac{4}{M}\right)\left(\mathrm{Var}(X_n) - \frac{M}{4}\right) - \left(1 - \frac{2}{M}\right)^{2n}\left(\frac{2\mathbb{E}[X_0]}{M} - 1\right)^2$$

$$v_{n+1} = av_n - cb^n$$

$$\sum_{n=0}^{\infty} v_{n+1}s^{n+1} = \sum_{n=0}^{\infty} av_n s^{n+1} - c\sum_{n=0}^{\infty} b^n s^{n+1}$$

$$g(s) - v_0 = asg(s) - \frac{cs}{1 - bs}$$

$$(1 - as)g(s) = v_0 - \frac{cs}{1 - bs}$$

$$g(s) = \frac{v_0}{1 - as} - \frac{cs}{(1 - bs)(1 - as)}$$

$$= \frac{v_0}{1 - as} - \frac{c}{b - a}\left(\frac{1}{1 - bs} - \frac{1}{1 - as}\right)$$

$b = (1 - \frac{2}{M})^2$, $a = 1 - 4/M$, $b - a = 4/M^2$.

$$
\begin{aligned}
g(s) &= \frac{v_0}{1-as} - \frac{c}{b-a}\left(\frac{1}{1-bs} - \frac{1}{1-as}\right) \\
&= \left(v_0 + \frac{c}{b-a}\right)\frac{1}{1-as} - \frac{c}{b-a}\frac{1}{1-bs} \\
&= \left(v_0 + \frac{c}{b-a}\right)\sum_{n=0}^{\infty} a^n s^n - \frac{c}{b-a}\sum_{n=0}^{\infty} b^n s^n
\end{aligned}
$$

$\frac{c}{b-a} = (M^2/4)\left(\frac{2\mathbb{E}[X_0]}{M} - 1\right)^2 = \left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2$ So

$$
\begin{aligned}
\mathrm{Var}(X_n) - \frac{M}{4} = &\left(\mathrm{Var}(X_0) - \frac{M}{4} + \left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2\right)\left(1 - \frac{4}{M}\right)^n \\
&- \underbrace{\left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2\left(1 - \frac{2}{M}\right)^{2n}}_{=(\mathbb{E}[X_n]-M/2)^2}
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{Var}(X_n) - \frac{M}{4} + (\mathbb{E}[X_n] - \frac{M}{2})^2 \\
=&\,\mathbb{E}(X_n^2) - (\mathbb{E}[X_n])^2 - \frac{M}{4} + (\mathbb{E}[X_n])^2 - M\mathbb{E}[X_n] + \frac{M^2}{4} \\
=&\,\mathbb{E}(X_n^2) - \frac{M}{4} - M\mathbb{E}[X_n] + \frac{M^2}{4} \\
=&\,\mathbb{E}(X_n(X_n - M)) + \frac{M(M-1)}{4}
\end{aligned}
$$

$$
X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M-X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}
$$

$$
X_{n+1}(X_{n+1} - M) = \begin{cases} (X_n + 1)(X_n - M + 1) & \text{w. p. } \frac{M-X_n}{M} \\ (X_n - 1)(X_n - M - 1) & \text{w. p. } \frac{X_n}{M} \end{cases}
$$

$$
= \begin{cases} X_n(X_n - M) + 1 - M + 2X_n & \text{w. p. } \frac{M-X_n}{M} \\ X_n(X_n - M) + 1 + M - 2X_n & \text{w. p. } \frac{X_n}{M} \end{cases}
$$

$$
\begin{aligned}
&\mathbb{E}(X_{n+1}(X_{n+1}-M)|X_n) \\
=\,& X_n(X_n-M)+1-(2X_n-M)^2/M \\
=\,& X_n(X_n-M)+1-(4X_n^2-4MX_n+M^2)/M \\
=\,& X_n(X_n-M)+1-4X_n^2/M-4X_n+M \\
=\,& X_n(X_n-M)+1-4X_n(X_n-M)/M+M \\
=\,& X_n(X_n-M)(1-4/M)+1+M
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(X_1) - \frac{M}{4} &= \left(\mathrm{Var}(X_0) - \frac{M}{4} + \left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2\right)\left(1 - \frac{4}{M}\right) \\
&\quad - \left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2 \left(1 - \frac{2}{M}\right)^2 \\
&= \left(\mathrm{Var}(X_0) - \frac{M}{4}\right)\left(1 - \frac{4}{M}\right) \\
&\quad + \left(\mathbb{E}[X_0] - \frac{M}{2}\right)^2 \underbrace{\left(1 - \frac{4}{M} - \left(1 - \frac{2}{M}\right)^2\right)}_{=-4/M^2} \\
&= \left(\mathrm{Var}(X_0) - \frac{M}{4}\right)\left(1 - \frac{4}{M}\right) - \left(\frac{2\mathbb{E}[X_0]}{M} - 1\right)^2
\end{aligned}
$$

Example 3: Branching Processes (Section 4.7)

Consider a population of individuals.

- All individuals have the same lifetime
- Each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the n-th generation, $n = 0, 1, 2, \ldots$.
If $X_{n-1} = k$, the k individuals in the $(n-1)$-th generation will independently produce $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,k}$ new offsprings, and $Z_{n,1}$, $Z_{n,2}$, ..., $Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j, \ j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i} \tag{1}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Mean of a Branching Process

Let $\mu = \mathbb{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$ be the mean # of offsprings produced by an individual. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$ and $Z_{n,i}$'s are i.i.d., we have

$$\mathbb{E}[X_n|X_{n-1}] = \mathbb{E}\left[\sum_{i=1}^{X_{n-1}} Z_{n,i} \middle| X_{n-1}\right] = X_{n-1}\mathbb{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]] = \mathbb{E}[X_{n-1}\mu] = \mu\mathbb{E}[X_{n-1}]$$

Then

$$\mathbb{E}[X_n] = \mu\mathbb{E}[X_{n-1}] = \mu^2\mathbb{E}[X_{n-2}] = \ldots = \mu^n\mathbb{E}[X_0]$$

- If $\mu < 1 \Rightarrow \mathbb{E}[X_n] \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \mathrm{P}(X_n \geq 1) = 0$ the branching processes will eventually die out.
- What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \mathrm{Var}[Z_{n,i}] = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$. $\mathrm{Var}(X_n)$ may be obtained using the conditional variance formula

$$\mathrm{Var}(X_n) = \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \mathrm{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]) = \mathrm{Var}(X_{n-1}\mu) = \mu^2\mathrm{Var}(X_{n-1})$$
$$\mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] = \sigma^2\mathbb{E}[X_{n-1}] = \sigma^2\mu^{n-1}\mathbb{E}[X_0].$$

Variance of a Branching Process

So

$$
\begin{aligned}
\mathrm{Var}(X_n) &= \sigma^2 \mu^{n-1} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-1}) \\
&= \sigma^2 \mu^{n-1} \mathbb{E}[X_0] + \mu^2 (\sigma^2 \mu^{n-2} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-2})) \\
&= \sigma^2 (\mu^{n-1} + \mu^n) \mathbb{E}[X_0] + \mu^4 \mathrm{Var}(X_{n-2}) \\
&= \sigma^2 (\mu^{n-1} + \mu^n) \mathbb{E}[X_0] + \mu^4 (\sigma^2 \mu^{n-3} \mathbb{E}[X_0] + \mu^2 \mathrm{Var}(X_{n-3})) \\
&= \sigma^2 (\mu^{n-1} + \mu^n + \mu^{n+1}) \mathbb{E}[X_0] + \mu^6 \mathrm{Var}(X_{n-3}) \\
&\ \ \vdots \\
&= \sigma^2 (\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}) \mathbb{E}[X_0] + \mu^{2n} \mathrm{Var}(X_0) \\
&= \begin{cases} \sigma^2 \mu^{n-1} \left(\frac{1-\mu^n}{1-\mu} \right) \mathbb{E}[X_0] + \mu^{2n} \mathrm{Var}(X_0) & \text{if } \mu \neq 1 \\ n\sigma^2 \mathbb{E}[X_0] + \mathrm{Var}(X_0) & \text{if } \mu = 1 \end{cases}
\end{aligned}
$$

4.5.1 The Gambler's Ruin Problem

- A gambler repeatedly plays a game until he goes bankrupt or his fortune reaches N.
- In each game, he can win \$1 with probability p or lose \$1 with probability $q = 1 - p$.
- Outcomes of different games are independent
- Define X_n = the gambler's fortune after the nth game.
- $\{X_n\}$ is a simple random walk w/ absorbing boundaries at 0 and N.

$$P_{00} = P_{NN} = 1,\ P_{i,i+1} = p, P_{i,i-1} = q,\ i = 1, 2, \ldots, N-1$$

- Two recurrent classes: $\{0\}$ and $\{N\}$
one transient class $\{1, 2, \ldots, N-1\}$
- Regardless of the initial fortune X_0, eventually $\lim_{n\to\infty} X_n = 0$ or N as all states are transient except 0 or N.

4.5.1 The Gambler's Ruin Problem

Denote A as the event that the gambler's fortune reaches N before reaches 0. Then

$$P_i = P(A|X_0 = i).$$

Conditioning on the outcome of the first game,

$$P_i = P(A|X_0 = i, \text{he wins the 1st game}) \underbrace{P(\text{he wins the 1st game})}_{=p}$$

$$+ P(A|X_0 = i, \text{he loses the 1st game}) \underbrace{P(\text{he loses the 1st game})}_{=q}$$

$$= P(A|X_0 = i, X_1 = i+1)p + P(A|X_0 = i, X_1 = i-1)q$$

$$= \underbrace{P(A|X_1 = i+1)}_{=P_{i+1}} p + \underbrace{P(A|X_1 = i-1)}_{=P_{i-1}} q \ (\because \text{ Markov})$$

We get a set of equations

$$P_i = pP_{i+1} + qP_{i-1} \quad \text{for } i = 1, 2, \ldots, N-1.$$
$$P_0 = 0, \quad P_N = 1$$

Solving the equations $P_i = pP_{i+1} + qP_{i-1}$

$$
\begin{aligned}
(p+q)P_i &= pP_{i+1} + qP_{i-1} \qquad \text{since } p+q=1 \\
\Leftrightarrow \quad q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\
\Leftrightarrow \quad P_{i+1} - P_i &= (q/p)(P_i - P_{i-1})
\end{aligned}
$$

As $P_0 = 0$,

$$
\begin{aligned}
P_2 - P_1 &= (q/p)(P_1 - P_0) = (q/p)P_1 \\
P_3 - P_2 &= (q/p)(P_2 - P_1) = (q/p)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= (q/p)(P_{i-1} - P_{i-2}) = (q/p)(q/p)^{i-2} P_1 = (q/p)^{i-1} P_1
\end{aligned}
$$

Adding up the equations above we get

$$
P_i - P_1 = \left[q/p + (q/p)^2 + \cdots + (q/p)^{i-1} \right] P_1
$$

Solving the equations $P_i = pP_{i+1} + qP_{i-1}$

$$
\begin{aligned}
(p+q)P_i &= pP_{i+1} + qP_{i-1} \qquad \text{since } p+q=1 \\
\Leftrightarrow \quad q(P_i - P_{i-1}) &= p(P_{i+1} - P_i) \\
\Leftrightarrow \quad P_{i+1} - P_i &= (q/p)(P_i - P_{i-1})
\end{aligned}
$$

As $P_0 = 0$,

$$
\begin{aligned}
P_2 - P_1 &= (q/p)(P_1 - P_0) = (q/p)P_1 \\
P_3 - P_2 &= (q/p)(P_2 - P_1) = (q/p)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= (q/p)(P_{i-1} - P_{i-2}) = (q/p)(q/p)^{i-2} P_1 = (q/p)^{i-1} P_1
\end{aligned}
$$

Adding up the equations above we get

$$P_i - P_1 = \left[q/p + (q/p)^2 + \cdots + (q/p)^{i-1}\right] P_1$$

From

$$P_i - P_1 = \left[q/p + (q/p)^2 + \cdots + (q/p)^{i-1}\right] P_1$$

we get

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)} P_1 & \text{if } p \neq q \\ i P_1 & \text{if } p = q \end{cases}$$

As $P_N = 1$, we get

$$P_1 = \begin{cases} \frac{1-(q/p)}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ 1/N & \text{if } p = 0.5 \end{cases}$$

So

$$P_i = \begin{cases} \frac{1-(q/p)^i}{1-(q/p)^N} & \text{if } p \neq 0.5 \\ i/N & \text{if } p = 0.5 \end{cases}$$

If the gambler will never quit with whatever fortune he has ($N = \infty$), then

$$\lim_{N\to\infty} P_i = \begin{cases} 1 - (q/p)^i & \text{if } p > 0.5 \\ 0 & \text{if } p \leq 0.5 \end{cases}$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space = $\{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{first time to reach state } j \text{ when starting from state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, N_{01}, $N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i+1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i-1 \end{cases} \tag{2}$$

Observe that $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i,i+1}$ is indep of $N^*_{i-1,i}$.

4.5.3 Random Walk w/ Reflective Boundary at 0 (Cont'd)

Let $m_i = \mathbb{E}(N_{i,i+1})$. Taking expected value on Equation (**??**), we get

$$m_i = \mathbb{E}[N_{i,i+1}] = 1 + q\mathbb{E}[N^*_{i-1,i}] + q\mathbb{E}[N^*_{i,i+1}] = 1 + q(m_{i-1} + m_i)$$

Rearrange terms we get $pm_i = 1 + qm_{i-1}$ or

$$\begin{aligned} m_i &= \frac{1}{p} + \frac{q}{p} m_{i-1} \\ &= \frac{1}{p} + \frac{q}{p}(\frac{1}{p} + \frac{q}{p} m_{i-2}) \\ &= \frac{1}{p}\left[1 + \frac{q}{p} + (\frac{q}{p})^2 + \ldots + (\frac{q}{p})^{i-1}\right] + (\frac{q}{p})^i m_0 \end{aligned}$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + (\frac{q}{p})^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Mean of $N_{0,n}$

Recall that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$

$$\begin{aligned}\mathbb{E}[N_{0n}] &= m_0 + m_1 + \ldots + m_{n-1} \\ &= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2}[1 - (\frac{q}{p})^n] & \text{if } p \neq 0.5 \\ n^2 & \text{if } p = 0.5 \end{cases}\end{aligned}$$

When

$$\begin{array}{lll} p > 0.5 & \mathbb{E}[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2} & \text{linear in } n \\ p = 0.5 & \mathbb{E}[N_{0n}] = n^2 & \text{quadratic in } n \\ p < 0.5 & \mathbb{E}[N_{0n}] = O(\frac{2pq}{(p-q)^2}(\frac{q}{p})^n) & \text{exponential in } n \end{array}$$

Exercise 4.50 on p.284

A Markov chain has transition probability matrix

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

Communicating classes:

$$\underset{\substack{\uparrow \\ \text{transient}}}{\{1,2\}} \qquad \underset{\substack{\uparrow \\ \text{recurrent}}}{\{3,4\}} \qquad \underset{\substack{\uparrow \\ \text{recurrent}}}{\{5,6\}}$$

Find $\lim_{n\to\infty} P^{(n)}$.

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is transient, hence,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

Observe that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ if j is NOT accessible from i

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & ? & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

The two classes $\{3,4\}$ and $\{5,6\}$ do not communicate and hence the transition probabilities in between are all 0.

Exercise 4.50 on p.284 (Cont'd)

Since the Markov chain restricted to the closed class {3,4} is also a Markov chain with the transition matrix

$$\begin{array}{c} \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{cc} 3 & 4 \end{array} \\ \begin{pmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{pmatrix} \end{array}$$

and the limiting distribution of a two-state Markov chain with the transition matrix $\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$ is $\left(\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}\right)$, we get

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & ? \\ 0 & 0 & 0 & 0 & ? & ? \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

For the same reason,

$$\lim_{n\to\infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & ? & ? & ? & ? \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{array}$$

Exercise 4.50 on p.284 (Cont'd)

It remains to find

$$\pi_{ij} = \lim_{n\to\infty} P_{ij}^{(n)}$$

from a transient state $i = 1, 2$ to a recurrent state $j = 3$, 4, 5, or 6.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0.2 & 0.4 & 0 & 0.3 & 0 & 0.1 \\ 0.1 & 0.3 & 0 & 0.4 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.7 & 0 & 0 \\ 0 & 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.2 & 0.8 \end{pmatrix} \end{array}$$

By the Chapman-Kolmogorov Equation,

$$\begin{aligned} P_{13}^{(n+1)} &= P_{11}P_{13}^{(n)} + P_{12}P_{23}^{(n)} + P_{13}P_{33}^{(n)} + P_{14}P_{43}^{(n)} + P_{15}P_{53}^{(n)} + P_{16}P_{63}^{(n)} \\ &= 0.2P_{13}^{(n)} + 0.4P_{23}^{(n)} + 0 + 0.3P_{43}^{(n)} + 0 + 0.1\underbrace{P_{63}^{(n)}}_{=0} \end{aligned}$$

where $P_{63}^{(n)} = 0$ since state 3 and 6 do not communicate.

Let $n \to \infty$ and recall we've shown earlier that $\lim_{n\to\infty} P_{43}^{(n)} = 6/13$. We get the equation

$$\pi_{13} = 0.2\pi_{13} + 0.4\pi_{23} + 0.3 \times \frac{6}{13}.$$

Exercise 4.50 on p.284 (Cont'd)

Similarly,

$$\begin{aligned} P_{23}^{(n+1)} &= P_{21}P_{13}^{(n)} + P_{22}P_{23}^{(n)} + P_{23}P_{33}^{(n)} + P_{24}P_{43}^{(n)} + P_{25}P_{53}^{(n)} + P_{26}P_{63}^{(n)} \\ &= 0.1P_{13}^{(n)} + 0.3P_{23}^{(n)} + 0 + 0.4P_{43}^{(n)} + 0 + 0.2\underbrace{P_{63}^{(n)}}_{=0} \end{aligned}$$

where $P_{63}^{(n)} = 0$ since state 3 and 6 do not communicate. Let $n \to \infty$ and recall we've shown earlier that $\lim_{n\to\infty} P_{43}^{(n)} = 6/13$. We get the equation

$$\pi_{23} = 0.1\pi_{13} + 0.3\pi_{23} + 0.4 \times \frac{6}{13}.$$

Along with the equation $\pi_{13} = 0.2\pi_{13} + 0.4\pi_{23} + 0.3 \times \frac{6}{13}$ obtained on the previous page, we get

$$\pi_{13} = \frac{37}{52} \times \frac{6}{13} = \frac{111}{338}, \quad \pi_{23} = \frac{35}{52} \times \frac{6}{13} = \frac{105}{338}$$

Exercise 4.50 on p.284 (Cont'd)

Similarly

$$P_{15}^{(n+1)} = P_{11}P_{15}^{(n)} + P_{12}P_{25}^{(n)} + P_{13}P_{35}^{(n)} + P_{14}P_{45}^{(n)} + P_{15}P_{55}^{(n)} + P_{16}P_{65}^{(n)}$$
$$= 0.2P_{15}^{(n)} + 0.4P_{25}^{(n)} + 0 + 0.3\underbrace{P_{45}^{(n)}}_{=0} + 0 + 0.1P_{65}^{(n)}$$

$$P_{25}^{(n+1)} = P_{21}P_{15}^{(n)} + P_{22}P_{25}^{(n)} + P_{23}P_{35}^{(n)} + P_{24}P_{45}^{(n)} + P_{25}P_{55}^{(n)} + P_{26}P_{65}^{(n)}$$
$$= 0.1P_{15}^{(n)} + 0.3P_{25}^{(n)} + 0 + 0.4\underbrace{P_{45}^{(n)}}_{=0} + 0 + 0.2P_{65}^{(n)}$$

where $P_{45}^{(n)} = 0$ since state 4 and 5 do not communicate. Letting $n \to \infty$ and since $\lim_{n\to\infty} P_{65}^{(n)} = 2/7$, we get the equations

$$\pi_{15} = 0.2\pi_{15} + 0.4\pi_{25} + 0.1(2/7)$$
$$\pi_{25} = 0.1\pi_{15} + 0.3\pi_{25} + 0.2(2/7)$$

and can find the solutions

$$\pi_{15} = \frac{15}{52} \times \frac{2}{7} = \frac{15}{182}, \quad \pi_{25} = \frac{17}{52} \times \frac{2}{7} = \frac{17}{182}.$$

Exercise 4.50 on p.284 (Cont'd)

One can use the same method to find that

$$\pi_{14} = \frac{37}{52} \times \frac{7}{13}, \quad \pi_{24} = \frac{35}{52} \times \frac{7}{13}$$
$$\pi_{16} = \frac{15}{52} \times \frac{5}{7}, \quad \pi_{26} = \frac{17}{52} \times \frac{5}{7}$$

Hence,

$$\lim_{n \to \infty} P^{(n)} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & \frac{37}{52} \times \frac{6}{13} & \frac{37}{52} \times \frac{7}{13} & \frac{15}{52} \times \frac{2}{7} & \frac{15}{52} \times \frac{5}{7} \\ 0 & 0 & \frac{35}{52} \times \frac{6}{13} & \frac{35}{52} \times \frac{7}{13} & \frac{17}{52} \times \frac{2}{7} & \frac{17}{52} \times \frac{5}{7} \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 6/13 & 7/13 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \\ 0 & 0 & 0 & 0 & 2/7 & 5/7 \end{pmatrix} \end{array}$$