STAT253/317 Lecture 25

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10.5 The Maximum of Brownian Motion with Drift (11th edition only, not in 10th edition)

Maximum of a Brownian Motion with drift

Let $\{X(t), t \geq 0\}$ be a Brownian Motion with drift coefficient μ and variance parameter σ^2. Consider the maximum of the process up to time t

$$M(t) = \max_{0 \leq s \leq t} X(s)$$

Also consider the hitting time to the value $a > 0$

$$T_a = \min\{t : X(t) = a\}.$$

- It remains true that $P(T_a < t) = P(M(t) \geq a)$.
- Recall for Brownian motion without drift, we use the Reflection principle to find $P(T_a < t)$
- Reflection principle doesn't apply to Brownian motion with drift. We need other tools.

Theorem 10.2

Let $X(t)$ be the Brownian motion process $\{B(t), t \geq 0\}$ with drift coefficient μ and variance parameter σ^2. Given that $X(t) = x$, the conditional distribution of $\{X(s) : 0 \leq s \leq t\}$ does not depend on the value of μ.

Proof. Given $X(t) = x$, $\{X(s) : 0 \leq s \leq t\}$ remains a Gaussian process. As a Gaussian process is uniquely determined by its mean function and the covariance function, it suffices to show that the mean function

$$m(s) = \mathbb{E}[X(s)|X(t) = x], \quad 0 \leq s \leq t$$

and covariance

$$C(s, u) = \mathrm{Cov}(X(s), X(u)|X(t) = x), \quad 0 \leq s, u \leq t$$

do not depend on the value of μ.

For jointly normal random variables, zero covariance implies independence. If we can find a scalar c such that

$$\mathrm{Cov}(X(s) - cX(t), X(t)) = 0,$$

then $X(s) - cX(t)$ and $X(t)$ would be indep.. The conditional distribution of of $X(s) - cX(t)$ given $X(t) = x$ would simply be its unconditional distribution

$$X(s) = c\underbrace{X(t)}_{x} + \underbrace{X(s) - cX(t)}_{\sim N(\mu s - c\mu t,\, \sigma^2(s-2cs+c^2t))}$$

$$\sim N\left(cx + \mu s - c\mu t, \sigma^2(s - 2cs + c^2 t)\right).$$

To make

$$\begin{aligned}\mathrm{Cov}(X(s) - cX(t), X(t)) &= \mathrm{Cov}(X(s), X(t)) - \mathrm{Cov}(cX(t), X(t)) \\ &= \sigma^2 s - c\sigma^2 t = \sigma^2(s - ct) = 0,\end{aligned}$$

we must let $c = s/t$. Thus given $X(t) = x$ for $s < t$,

$$X(s) \sim N\left(\frac{sx}{t} + \underbrace{\mu s - (s/t)\mu t}_{=\mu s - \mu s = 0}, \sigma^2 \frac{s(t-s)}{t}\right) = N\left(\frac{sx}{t},\ \sigma^2 \frac{s(t-s)}{t}\right).$$

So the mean function $m(s) = \mathbb{E}[X(s)|X(t) = x] = \frac{sx}{t}$ and the covariance function $C(s, u) = \mathrm{Cov}(X(s), X(u)|X(t) = x) = \sigma^2 \frac{s(t-s)}{t}$ don't depend on the drift coefficient μ.

Theorem 10.3 on p.626-627

$$P(M(t) \geq y | X(t) = x) = \begin{cases} 1 & \text{if } x \geq y \geq 0 \\ e^{-2y(y-x)/t\sigma^2} & \text{if } x < y, y \geq 0 \end{cases}$$

Proof.

- The equality is trivial when $x \geq y$ as $M(t) \geq X(t) = x \geq y$.
- When $x < y$, as Theorem 10.2 implies the conditional distribution of $M(t) = \max_{0 \leq s \leq t} X(s)$ given $X(t) = x$ is identical for all values of μ, we just need to show the identity for the case with drift $\mu = 0$, to which the Reflection Principle is applicable.
- For $h > 0$ small enough that $y - x - h > 0$, by the Reflection Principle,

$$\begin{aligned} & P(M(t) \geq y, x \leq X(t) \leq x + h) \\ = & P(M(t) \geq y, 2y - x - h \leq X(t) \leq 2y - x) \\ = & P(2y - x - h \leq X(t) \leq 2y - x) \end{aligned}$$

  where the last equality is valid since $M(t) \geq X(t) \geq 2y - x - h > y$ as $y - x - h > 0$.

$$P(M(t) \geq y | X(t) = x) = \lim_{h \to 0} \frac{P(M(t) \geq y, x \leq X(t) \leq x + h)}{P(x \leq X(t) \leq x + h)}$$
$$= \lim_{h \to 0} \frac{P(2y - x - h \leq X(t) \leq 2y - x)}{P(x \leq X(t) \leq x + h)}$$
$$= \frac{f(2y - x)}{f(x)}$$

where

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right)$$

is the density function of $X(t) \sim N(0, \sigma^2 t)$ with drift $\mu = 0$. So

$$P(M(t) \geq y | X(t) = x) = \frac{f(2y - x)}{f(x)} = \frac{\exp(-(2y - x)^2/(2\sigma^2 t))}{\exp(-x^2/(2\sigma^2 t))}$$
$$= \exp\left(-\frac{(2y - x)^2 - x^2}{2\sigma^2 t}\right) = e^{-\frac{2y(y-x)}{t\sigma^2}}$$

Corollary 10.1 on p.627-628

Conditioning on $X(t)$ and using Theorem 10.3 yields

$$P(M(t) \geq y) = \int_{-\infty}^{\infty} P(M(t) \geq y | X(t) = x) f_{X(t)}(x) dx$$
$$= \int_{-\infty}^{y} \underbrace{e^{-\frac{2y(y-x)}{t\sigma^2}} f_{X(t)}(x)}_{\text{see below}} dx + \underbrace{\int_{y}^{\infty} 1 \cdot f_{X(t)}(x) dx}_{=P(X(t)>y)}$$

$$e^{-\frac{2y(y-x)}{t\sigma^2}} f_{X(t)}(x) = e^{-\frac{2y(y-x)}{t\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2 + 4y(y-x)}{2\sigma^2 t}}$$

in which

$$(x-\mu t)^2 + 4y(y-x) = x^2 - 2\mu t x + \mu^2 t^2 + 4y^2 - 4xy$$
$$= x^2 - 2(\mu t + y)x + \underbrace{\mu^2 t^2 \overbrace{+4\mu t y}^{\text{add a term}} + 4y^2}_{=(\mu t + 2y)^2} \overbrace{-4\mu t y}^{\text{subtract a term}}$$
$$= \underbrace{x^2 - 2(\mu t + y)x + (\mu t + 2y)^2}_{=(x-(\mu t+2y))^2} - 4\mu t y$$

Corollary 10.1 on p.627-628 (Cont'd)

Putting everything together, we get

$$P(M(t) \geq y) = e^{\frac{2\mu t y}{\sigma^2 t}} \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-(\mu t+2y))^2}{2\sigma^2 t}} dx + P(X(t) > y)$$

Making the change of variable $u = x - 2y$ gives

$$P(M(t) \geq y) = e^{\frac{2\mu y t}{\sigma^2 t}} \int_{-\infty}^{-y} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(u-\mu t)^2}{2\sigma^2 t}}}_{\text{density of } X(t)} du + P(X(t) > y)$$

$$= e^{\frac{2\mu y}{\sigma^2}} P(X(t) < -y) + P(X(t) > y)$$

$$= e^{\frac{2\mu y}{\sigma^2}} \Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right) + 1 - \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right)$$

since $X(t) \sim N(\mu t, \sigma^2 t)$.

Note that for $\mu = 0$, we get
$P(M(t) \geq y) = P(X(t) < -y) + P(X(t) > y) = P(|X(t)| > y)$,
which agrees with our calculation before.

Hitting Time for Brownian Motion with drift

Also consider the hitting time to the value $y > 0$

$$T_y = \min\{t : X(t) = y\}.$$

It remains true that $T_y < t$ if and only if $M(t) \geq y$. So

$$P(T_y < t) = e^{\frac{2\mu y}{\sigma^2}} \Phi\left(\frac{-y - \mu t}{\sigma\sqrt{t}}\right) + 1 - \Phi\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right)$$