STAT253/317 Winter 2021 Lecture 24

Yibi Huang

- Brownian Motion with Drift
- Stopping Time, Strong Markov Property (Review)
- Wald's Identities for Brownian Motion

Brownian Motion with Drift

A stochastic process $\{B(t), t \geq 0\}$ is said to be a *Brownian motion process with drift coefficient μ and variance parameter σ^2* if

(i) $B(0) = 0$;

(ii) $\{B(t), t \geq 0\}$ has stationary and independent increments;

(iii) for every $t \geq 0, s \geq 0$, $B(t+s) - B(s) \sim N(\mu t, \sigma^2 t)$

Stopping Time (Review)

For a continuous time stochastic process $\{X(t), t \geq 0\}$, a *stopping time T with respect to* $\{X(t), t \geq 0\}$ is a nonnegative random variable, such that the event $\{T \leq t\}$ depends only on $\{X(s), 0 \leq s \leq t\}$ ~~but not $\{X(s), s > t\}$~~.

Remark: If T is a stopping time with respect to $\{X(t), t \geq 0\}$, for each non-random $n > 0$, the stopping time truncated at n

$$(T \wedge n) \text{ defined as } \min(T, n)$$

is also a stopping time with respective to $\{X(t), t \geq 0\}$.
Reason: $\{(T \wedge n) \leq t\} = \{T \leq t\} \cup \{n \leq t\}$

- The event $\{n \leq t\}$ is non-random, does not depend on $\{X(s)\}$
- The event $\{T \leq t\}$ depends only on $\{X(s), 0 \leq s \leq t\}$ but not $\{X(s), s > t\}$ since T is a stopping time

Hence the event $\{(T \wedge n) \leq t\}$ depends on $\{X(s), 0 \leq s \leq t\}$ only but not $\{X(s), s > t\}$, which shows $(T \wedge n)$ is also a stopping time.

Strong Markov Property (Review)

Let $\{B(t), t \geq 0\}$ be a Brownian Motion (with drift μ), and let T be a stopping time respective to $\{B(t), t \geq 0\}$. Then

(a) Define $Z(t) = B(t + T) - B(T)$, $t \geq 0$.
Then $\{Z(t), t \geq 0\}$ is also a Brownian Motion with drift μ

(b) For each $t > 0$, $\{Z(s), 0 \leq s \leq t\}$ is independent of $\{B(s), 0 \leq s \leq T\}$

Remark: If T is not a stopping time, the Strong Markov Property may not be true. For example, let

$$T = T_{\max} = \min\left\{t : B(t) = \max_{0 \leq s \leq 1} B(s)\right\},$$

where $\{B(t), t \geq 0\}$ is a standard Brownian motion.

- $T_{\max}$ is not a stopping time since the event $\{T_{\max} \leq t\}$ depends not just $\{B(s), 0 \leq s \leq t\}$, but on the entire $\{B(s), 0 \leq s \leq 1\}$.
- Since $B(T_{\max})$ will be the maximum of $\{B(s), 0 \leq s \leq 1\}$, $B(t + T_{\max}) - B(T_{\max})$ will be ≤ 0 for $t \leq 1 - T_{\max}$, and hence is not Brownian motion

Wald's Identities for Brownian Motion

If $\{B(t), t \geq 0\}$ is a Brownian motion process with drift μ and variance parameter σ^2, and T is a **bounded stopping time** with respect to $\{B(t)\}$, then

(i) $\mathbb{E}[B(T)] = \mu\mathbb{E}[T]$,

(ii) $\mathbb{E}[B^2(T)] = \sigma^2\mathbb{E}[T] + \mu^2\mathbb{E}[T^2]$,

(iii) $\mathbb{E}[e^{\theta B(T) - (\theta\mu + \frac{\theta^2\sigma^2}{2})T}] = 1$ for all $\theta \in \mathbb{R}$

Remark:

- For *nonrandom* times $T = t$, the identities follows from the elementary properties of the normal distribution
- If T is *unbounded*, the identities may not be true
  - Example: if $T = T_1$ be the hitting time to value 1 of a standard Brownian motion, then $B(T) = 1$. So $\mathbb{E}[B(T)] \neq 0$.
- If T is not a stopping time, the identities may also fail.
  - Example: if $T = T_{\max} = \min\{t : B(t) = \max_{0 \leq s \leq 1} B(s)\}$ then $\mathbb{E}[B(T_{\max})] = \mathbb{E}[\max_{0 \leq s \leq 1} B(s)] > 0$.

Application of Wald's Identities

For constants $a, b > 0$ Let $T = T_{-a,b}$ be the first time t such that the standard Brownian Motion process hit $-a$ or b

$$T_{-a,b} = \min\{t : B(t) = -a, \text{ or } B(t) = b\}$$

- T is a stopping time since the event

  $$\{T \leq t\} = \left\{\max_{0 \leq s \leq t} B(s) \geq b\right\} \bigcup \left\{\min_{0 \leq s \leq t} B(s) \leq -a\right\},$$

  depends on $\{B(s), 0 \leq s \leq t\}$ only.
- T is finite, but <u>unbounded</u> $\Rightarrow$ Wald's identities may <u>not</u> apply.
- However, for each integer $n \geq 1$, the random variable $T \wedge n = \min(T, n)$ is a bounded stopping time.
  By the first and second Wald's identities, we have

  $$\mathbb{E}[B(T \wedge n)] = 0 \quad \text{and} \quad \mathbb{E}[B^2(T \wedge n)] = \mathbb{E}[T \wedge n]$$

Application of Wald's Identities (Cont'd)

- From that $-a \le B(T \wedge n) \le b$, we know $|B(T \wedge n)|$ is uniformly bounded by $a + b$ for all n
- As $\mathrm{P}(T < \infty) = 1$, we have $\lim_{n\to\infty} B(T \wedge n) = B(T)$ w/ prob. 1.
- By Bounded Convergence Theorem,

$$\mathbb{E}[B(T)] = \lim_{n\to\infty} \mathbb{E}[B(T \wedge n)] = 0 \tag{1}$$

$$\mathbb{E}[B^2(T)] = \lim_{n\to\infty} \mathbb{E}[B^2(T \wedge n)] = \lim_{n\to\infty} \mathbb{E}[T \wedge n] = \mathbb{E}[T] \tag{2}$$

- Because $B(T) = -a$ or b, from that

$$\mathbb{E}[B(T)] = -a\mathrm{P}(B(T) = -a) + b\mathrm{P}(B(T) = b) = 0$$

and that $\mathrm{P}(B(T) = -a) + \mathrm{P}(B(T) = b) = 1$, it follows that

$$\mathrm{P}(B(T) = -a) = \frac{b}{a+b}, \quad \mathrm{P}(B(T) = b) = \frac{a}{a+b}$$

- From the above and (2), one may easily deduce that

$$\mathbb{E}[T] = \mathbb{E}[B^2(T)] = a^2\mathrm{P}(B(T) = -a) + b^2\mathrm{P}(B(T) = b) = ab$$

Exercise 10.22: $T_{-a,b}$ for Brownian with Drift

Let $\{B(t), t \geq 0\}$ be Brownian Motion with drift coefficient $\mu \neq 0$ and variance parameter σ^2. For constants $a, b > 0$ let

$$T = T_{-a,b} = \min\{t : B(t) = -a, \text{ or } B(t) = b\}$$

T is again a finite but <u>unbounded</u> stopping time, so Wald's identities may <u>not</u> be applied directly. However, using the truncated stopping time $T \wedge n = \min(T, n)$ and Bounded Convergence Theorem, we can prove that the first Wald's identity holds for T

$$\mu\mathbb{E}[T] = \mathbb{E}[B(T)] = -a\mathrm{P}(B(T) = -a) + b\mathrm{P}(B(T) = b).$$

However, when $\mu \neq 0$, we cannot use this equation and that $\mathrm{P}(B(T) = -a) + \mathrm{P}(B(T) = b) = 1$ to solve for $\mathrm{P}(B(T) = -a)$ and $\mathrm{P}(B(T) = b)$ since $\mathbb{E}[T]$ is unknown. Instead we will use the third Wald's identity.

Exercise 10.22: $T_{-a,b}$ for Brownian with Drift (Cont'd)

- By the third Wald's identity, we have

$$\mathbb{E}[e^{\theta B(T\wedge n)-(\theta\mu+\frac{\theta^2\sigma^2}{2})(T\wedge n)}] = 1 \quad \text{for all } \theta \in \mathbb{R}. \qquad (3)$$

- Let us choose $\theta = \theta_0 = -2\mu/\sigma^2$ so that the 2nd term in the exponent of (3) vanishes. So

$$\mathbb{E}[e^{\theta_0 B(T\wedge n)}] = 1$$

- $-a \leq B(T \wedge n) \leq b \Rightarrow |B(T \wedge n)| \leq a + b$

$$\Rightarrow e^{\theta_0 B(T\wedge n)} \leq e^{\theta_0(a+b)}$$

- By the Bounded Convergence Theorem,

$$1 = \lim_{n\to\infty} \mathbb{E}[e^{\theta_0 B(T\wedge n)}] = \mathbb{E}[e^{\theta_0 B(T)}]$$
$$= e^{-\theta_0 a}\mathrm{P}(B(T) = -a) + e^{\theta_0 b}\mathrm{P}(B(T) = b)$$

Exercise 10.22: $T_{-a,b}$ for Brownian with Drift (Cont'd)

Solving the equation

$$1 = e^{-\theta_0 a}\mathrm{P}(B(T) = -a) + e^{\theta_0 b}\mathrm{P}(B(T) = b)$$

and the equation $\mathrm{P}(B(T) = -a) + \mathrm{P}(B(T) = b) = 1$ for $\mathrm{P}(B(T) = -a)$ and $\mathrm{P}(B(T) = b)$, one can get that

$$\mathrm{P}(B(T) = -a) = \frac{1 - e^{\theta_0 b}}{e^{-\theta_0 a} - e^{\theta_0 b}}, \quad \mathrm{P}(B(T) = b) = \frac{e^{-\theta_0 a} - 1}{e^{-\theta_0 a} - e^{\theta_0 b}}$$

Theorem 1. Let $\{B(t), t \geq 0\}$ be a Brownian Motion with drift coefficient $\mu \neq 0$ and variance parameter σ^2, the probability that the process reach $b > 0$ before hitting $-a < 0$ is given by

$$\mathrm{P}(B(T_{-a,b}) = b) = \frac{\exp(2\mu a/\sigma^2) - 1}{\exp(2\mu a/\sigma^2) - \exp(-2\mu b/\sigma^2)}$$

Proof of Wald's Identities for Brownian Motion

- Since T is bounded, there is a nonrandom $N < \infty$ such that $\mathrm{P}(T < N) = 1$
- By the Strong Markov Property, the post-T process $B(t + T) - B(T)$ is
  - also a Brownian Motion process with drift μ and variance parameter σ^2, and
  - independent of $\{B(s), 0 \le s \le T\}$, and in particular, independent of the random vector $(T, B(T))$.
- Hence, given that $T = s$ the conditional distribution of $B(N) - B(T)$ is normal with mean $\mu(N - s)$ and variance $\sigma^2(N - s)$. It follows that

$$\mathbb{E}\left[e^{\theta[B(N)-B(T)]-\theta\mu(N-T)-\frac{\theta^2\sigma^2(N-T)}{2}}\,\middle|\,T, B(T)\right] = 1$$

Proof of Wald's Identities (Cont'd)

Therefore

$$
\begin{aligned}
&\mathbb{E}[e^{\theta B(T)-\theta\mu T-\frac{\theta^2\sigma^2 T}{2}}] = \mathbb{E}[e^{\theta B(T)-\theta\mu T-\frac{\theta^2\sigma^2 T}{2}}] \times 1 \\
&= \mathbb{E}[e^{\theta B(T)-\theta\mu T-\frac{\theta^2\sigma^2 T}{2}}] \\
&\qquad \times \mathbb{E}\left[e^{\theta[B(N)-B(T)]-\theta\mu(N-T)-\frac{\theta^2\sigma^2(N-T)}{2}}\,\middle|\,T, B(T)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\theta B(T)-\theta\mu T-\frac{\theta^2\sigma^2 T}{2}+\theta[B(N)-B(T)]-\theta\mu(N-T)-\frac{\theta^2\sigma^2(N-T)}{2}}\,\middle|\,T, B(T)\right]\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{\theta B(N)-\theta\mu N-\frac{\theta^2\sigma^2 N}{2}}\,\middle|\,T, B(T)\right]\right] \\
&= \mathbb{E}[e^{\theta B(N)-\theta\mu N-\frac{\theta^2\sigma^2 N}{2}}] = 1
\end{aligned}
$$

This proves the third identity.

The first and second identity can be derived by differentiating the third identity with respective to θ once and twice respectively, and letting $\theta = 0$.