STAT253/317 Winter 2019 Lecture 22&23

Yibi Huang

Chapter 10 Brownian Motion

- Brownian Motion as a Limit of Random Walk
- Brownian Motion as a Gaussian Process

10.2 Hitting Time, Maximum Value, Reflection Principle

Generalized Random Walk

The symmetric simple random walk $\{Y_n, n \geq 1\}$ can be defined alternatively as a sum of i.i.d. random variables

$$Y_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1$$

where X_i's are i.i.d. with distribution

$$X_i = \begin{cases} 1 & \text{w/ prob. } 0.5 \\ -1 & \text{w/ prob. } 0.5 \end{cases}$$

Generally, for any sequence of i.i.d random variables $X_1, X_2, \dots$ from an arbitrary distribution with $\mathbb{E}[X_i] = 0$, $\mathrm{Var}(X_i) = \sigma^2$, the partial sum process

$$Y_n = X_1 + X_2 + \cdots + X_n, \quad n \geq 1$$

is also called a **(generalized) random walk**.

10.1 Brownian Motion as a Limit of Random Walk

The Browian motion is in fact a limit of rescaled generalized random walk.

Let $X_1, X_2, \ldots$ be i.i.d. random variables, $\mathbb{E}[X_i] = 0$, $\mathrm{Var}(X_i) = \sigma^2$. Define

$$X(t) = \Delta x(X_1 + \ldots + X_{\lfloor t/\Delta t \rfloor})$$

where $\lfloor t/\Delta t \rfloor$ is the integer part of $t/\Delta t$.

We'd like to find the limit of $X(t)$ as $\underline{\Delta t \text{ and } \Delta x \text{ both} \to 0}$.

Observe

$$\mathbb{E}[X(t)] = 0, \quad \mathrm{Var}(X(t)) = \sigma^2(\Delta x)^2 \left\lfloor \frac{t}{\Delta t} \right\rfloor,$$

To have a non-trivial limit, Δt and Δx must maintain the relationship

$$\Delta t = c(\Delta x)^2.$$

as they approach 0. Let's take $c = 1$. In this case, as $\Delta t \to 0$, $\Delta x \to 0$, and $\Delta t = (\Delta x)^2$, we have

$$\mathbb{E}[X(t)] = 0, \quad \mathrm{Var}(X(t)) \to \sigma^2 t,$$

Moreover, since $\Delta x = \sqrt{\Delta t}$, by CLT

$$X(t) = \Delta x(X_1 + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor}) \approx \sqrt{t}\sigma \frac{X_1 + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor}}{\sqrt{\lfloor t/\Delta t \rfloor}\sigma} \to N(0, \sigma^2 t)$$

in distribution.

Observe that the discrete-time process

$$\{X(t),\ t = n\Delta t,\ n = 0, 1, 2 \ldots\}$$

has *independent* and *stationary increments* since

$$\begin{aligned} X(s) &= \Delta x(X_1 + \ldots + X_{\lfloor \frac{s}{\Delta t} \rfloor}), \text{ and} \\ X(t) - X(s) &= \Delta x(X_{\lfloor \frac{s}{\Delta t} \rfloor + 1} + \ldots + X_{\lfloor \frac{t}{\Delta t} \rfloor}) \end{aligned}$$

are independent, and for $t = l\Delta t > s = m\Delta t$, the distribution of $X(t) - X(s)$ depends on the number of terms $\lfloor \frac{t}{\Delta t} \rfloor - \lfloor \frac{s}{\Delta t} \rfloor$ $= (l - m) = (t - s)/(\Delta t)$ in the sum, but not s.

Thus the limit of $X(t)$ is a process with **independent** and **stationary increments**.

Definition of a Brownian Motion

Definition 1 A stochastic process $\{B(t), t \geq 0\}$ is said to be a Brownian Motion if

(i) $B(0) = 0$;

(ii) $\{B(t), t \geq 0\}$ has stationary and independent increments;

(iii) for every $t, s > 0$, $B(t+s) - B(s) \sim N(0, \sigma^2 t)$

A Brownian motion with $\sigma = 1$ is called a *standard Brownian motion process*

In fact, we can show that, as a function of t, the path of $B(t)$ is **continuous** w/ prob. 1.

Covariance Function of a Brownian Motion

For $t > s$

$$\begin{aligned}\mathrm{Cov}[B(t), B(s)] &= \mathrm{Cov}[B(t) - B(s) + B(s), B(s)] \\ &= \mathrm{Cov}[B(t) - B(s), B(s)] + \mathrm{Cov}[B(s), B(s)] \\ &= 0 + \mathrm{Var}[B(s)] \quad \text{(by indep. increment)} \\ &= \sigma^2 s\end{aligned}$$

The function

$$C(s, t) = \mathrm{Cov}(B(t), B(s)) = \sigma^2 \min(s, t)$$

is called the **covariance function** of the Brownian motion process.

10.6 Gaussian Processes

Definition 10.2. A stochastic process $\{X(t), t \geq 0\}$ is called a *Gaussian process* if $X(t_1), \ldots, X(t_n)$ has a multivariate normal distribution for all $t_1, \ldots, t_n$.

Because a multivariate normal distribution is completely determined by the marginal mean values and the covariance values it follows that the properties of a Gaussian process is completely determined by its *mean function*

$$m(t) = \mathbb{E}[X(t)]$$

and *covariance function*

$$C(s, t) = \mathrm{Cov}(X(s), X(t)).$$

That is, two Gaussian processes are the same if

their **mean functions** and **covariance functions** are identical.

Brownian Motion as a Gaussian Process

Alternatively, a Brownian motion can be defined as a Gaussian process with mean function

$$m(t) = \mathbb{E}[B(t)] = 0$$

and covariance function

$$C(s,t) = \mathrm{Cov}(B(s), B(t)) = \sigma^2 \min(s,t).$$

Properties of a Brownian Motion

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. One can prove each of the following processes below is also a standard Brownian motion by showing they are all Gaussian processes with the same mean function and covariance function as the standard Brownian motion.

(i) $\{-B(t), t \geq 0\}$ (ii) $\{B(t+s) - B(s), t \geq 0\}$
(iii) $\{aB(t/a^2), t \geq 0\}$ (iv) $\{tB(1/t), t \geq 0\}$

Properties of a Brownian Motion (Proofs)

We'll prove (iv) only. The proofs for the rest are similar. Clearly $\{tB(1/t), t \geq 0\}$ is a Gaussian process since it is a linear function of a Brownian motion process.

$$
\begin{aligned}
\mathbb{E}[tB(1/t)] &= t\mathbb{E}[B(1/t)] = 0 \quad \text{since } B(1/t) \sim N(0, 1/t) \\
\mathrm{Cov}[tB(1/t), sB(1/s)] &= ts\mathrm{Cov}[B(1/t), B(1/s)] \\
&= ts \min\left(\frac{1}{t}, \frac{1}{s}\right) = \begin{cases} ts(1/t) = s & \text{if } t > s \\ ts(1/s) = t & \text{if } t \leq s \end{cases} \\
&= \min(s, t)
\end{aligned}
$$

As the Gaussian process $\{tB(1/t), t \geq 0\}$ has the same mean function and variance function as a standard Brownian motion, it is also a standard Brownian motion.

Conditional Distribution

Given $B(t) = x$, what is the conditional distribution of $B(s)$?

If $t < s$, since Brownian motion has independent increments, $B(s) - B(t)$ is independent of $B(t)$, and hence given $B(t) = x$, the condition distribution of $B(s) - B(t)$ is the same as its unconditional distribution.

$$\begin{aligned}(B(s)|_{B(t)=x}) &= B(t) + [B(s) - B(t)] \\ &= x + \underbrace{B(s) - B(t)}_{\sim N(0,\sigma^2(s-t))} \\ &\sim N(x, \sigma^2(s-t)).\end{aligned}$$

What if $s < t$?

If we can find a scalar c such that $\mathrm{Cov}(B(s) - cB(t), B(t)) = 0$, then

$$B(s) - cB(t) \text{ and } B(t) \text{ are independent.}$$

Thus the conditional distribution of of $B(s) - cB(t)$ given $B(t)$ is the same as its unconditional distribution $N(0, \sigma^2(s - 2cs + c^2t))$. Given $B(t) = x$,

$$B(s) = c\underbrace{B(t)}_{x} + \underbrace{B(s) - cB(t)}_{\sim N(0,\sigma^2(s-2cs+c^2t))} \sim N\left(cx, \sigma^2(s - 2cs + c^2t)\right).$$

Because

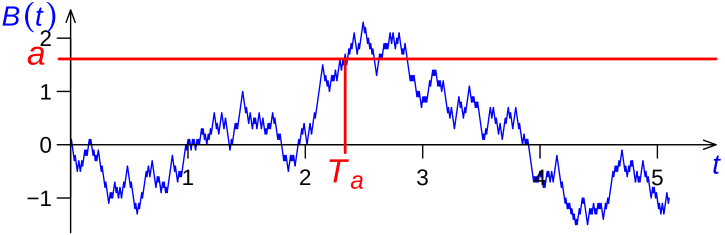
$$\begin{aligned}\mathrm{Cov}(B(s) - cB(t), B(t)) &= \mathrm{Cov}(B(s), B(t)) - \mathrm{Cov}(cB(t), B(t)) \\ &= \sigma^2 s - c\sigma^2 t = \sigma^2(s - ct)\end{aligned}$$

we know $c = s/t$. Thus the conditional distribution of $B(s)$ given $B(t) = x$ for $s < t$ is

$$N\left(\frac{sx}{t}, \sigma^2 \frac{s(t-s)}{t}\right).$$

Hitting Times (First Passage Times)

Let $T_a = \min\{t : B(t) = a\}$ be the first time the standard Brownian motion process hits a.



For $a > 0$, consider

$$\begin{aligned} \mathrm{P}(B(t) \geq a) = & \ \mathrm{P}(B(t) \geq a | T_a \leq t)\mathrm{P}(T_a \leq t) \\ & + \underbrace{\mathrm{P}(B(t) \geq a | T_a > t)}_{=0}\mathrm{P}(T_a > t) \end{aligned}$$

The 2nd term on the right is clearly 0, since by continuity, the process value cannot be $> a$ without having yet hit a.

For the 1st term, note if $T_a \leq t$, then the process hits a at some point in $[0, t]$ and, by symmetry, it is just as likely to be above or below a at time t. That is

$$\mathrm{P}(B(t) \geq a | T_a \leq t) = \frac{1}{2}$$

Thus

$$\mathrm{P}(T_a \leq t) = 2\mathrm{P}(B(t) \geq a) = 2 - 2\Phi(a/\sqrt{t}),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ is the CDF of $N(0, 1)$.
By symmetry, T_{-a} and T_a are identically distributed. Hence

$$\mathrm{P}(T_a \leq t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^{\infty} e^{-y^2/2} dy.$$

HW: Show that $\mathrm{P}(T_a < \infty) = 1$ and $\mathbb{E}[T_a] = \infty$ for $a > 0$.

Maximum

Another random variable of interest is

$$\max_{0 \le s \le t} B(s).$$

By the continuity of Brownian motion, we know

$$\max_{0 \le s \le t} B(s) \ge a \quad \Leftrightarrow \quad T_a \le t$$

Thus the distribution of for $\max_{0 \le s \le t} B(s)$ can be derived via T_a. For $a > 0$

$$\begin{aligned} \mathrm{P}\left(\max_{0 \le s \le t} B(s) \ge a\right) &= \mathrm{P}(T_a \le t) \\ &= 2\mathrm{P}(B(t) \ge a) = \mathrm{P}(|B(t)| \ge a) \\ &= 2 - 2\Phi(a/\sqrt{t}) \end{aligned}$$

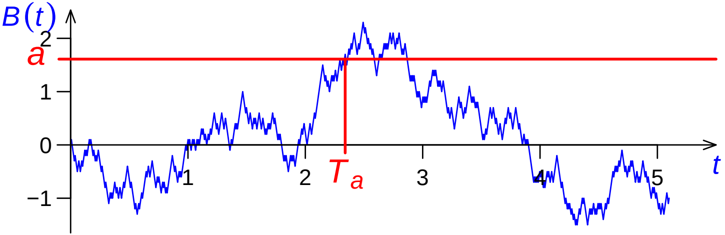
Note this means $\max_{0 \le s \le t} B(s)$ have the same distribution as $|B(t)|$.

Stopping Time

For a continuous time stochastic process $\{X(t), t \geq 0\}$, a *stopping time T with respect to* $\{X(t), t \geq 0\}$ is a nonnegative random variable, such that the event $\{T \leq t\}$ depends only on $\{X(s), 0 \leq s \leq t\}$.

Example

The hitting time $T_a = \min\{t : B(t) = a\}$ is a stopping time since the event $\{T_a \leq t\}$ is identical to the event $\left\{ \max_{0 \leq s \leq t} B(s) \geq a \right\}$

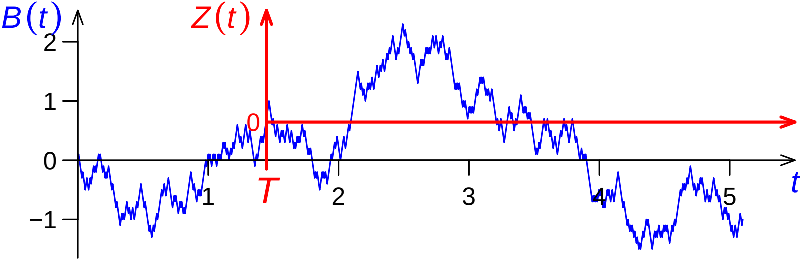


Strong Markov Property

Let $\{B(t), t \geq 0\}$ be a standard Brownian Motion, and let T be a stopping time respective to $\{B(t), t \geq 0\}$. Then

(a) Define $Z(t) = B(t+T) - B(T)$, $t \geq 0$.
Then $\{Z(t), t \geq 0\}$ is also a standard Brownian Motion

(b) For each $t > 0$, $\{Z(s), 0 \leq s \leq t\}$ is independent of $\{B(u), 0 \leq u \leq T\}$

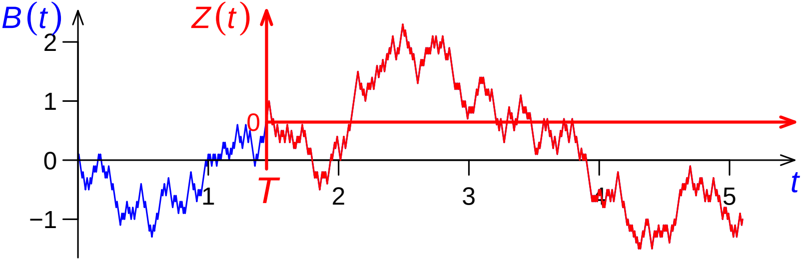


Strong Markov Property

Let $\{B(t), t \geq 0\}$ be a standard Brownian Motion, and let T be a stopping time respective to $\{B(t), t \geq 0\}$. Then

(a) Define $Z(t) = B(t + T) - B(T)$, $t \geq 0$.
Then $\{Z(t), t \geq 0\}$ is also a standard Brownian Motion

(b) For each $t > 0$, $\{Z(s), 0 \leq s \leq t\}$ is independent of $\{B(u), 0 \leq u \leq T\}$

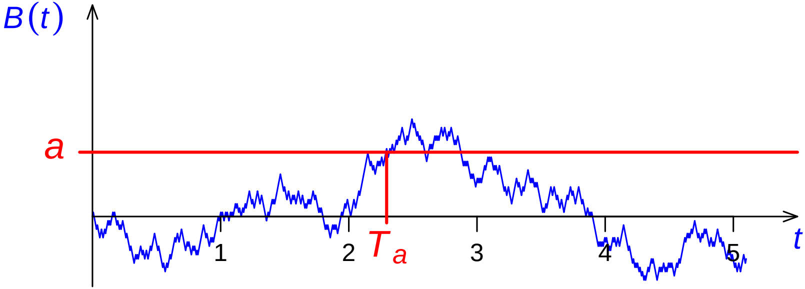


Reflection Principle

Let T_a be the first passage time to the value a of a standard Brownian Motion $\{B(t), t \geq 0\}$. Define a new process

$$\overline{B}(t) = \begin{cases} B(t) & \text{for } t \leq T_a \\ 2a - B(t) & \text{for } t > T_a \end{cases}$$

Then $\{\overline{B}(t), t \geq 0\}$ is also a standard Brownian Motion.

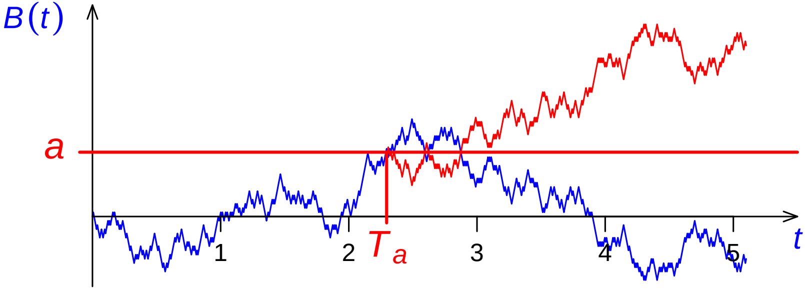


Reflection Principle

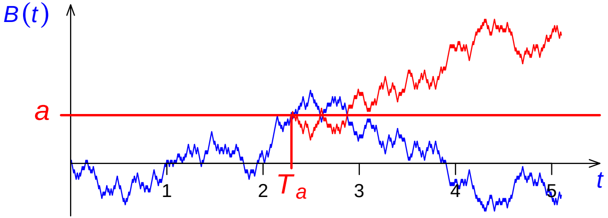
Let T_a be the first passage time to the value a of a standard Brownian Motion $\{B(t), t \geq 0\}$. Define a new process

$$\overline{B}(t) = \begin{cases} B(t) & \text{for } t \leq T_a \\ 2a - B(t) & \text{for } t > T_a \end{cases}$$

Then $\{\overline{B}(t), t \geq 0\}$ is also a standard Brownian Motion.



Proof of the Reflection Principle



For $t > T_a$, note

$$B(t) = a + B(t) - a = B(T_a) + B(t) - B(T_a).$$

- By Strong Markov Property, $B(s + T_a) - B(T_a) = B(s + T_a) - a$ is also a Brownian Motion, independent of $\{B(s), 0 \leq s \leq T_a\}$.
- Also note that if $\{B(t), t \geq 0\}$ is a standard Brownian motion, so is $\{-B(t), t \geq 0\}$. Hence $\{a - B(s + T_a), s \geq 0\}$ is also a Brownian Motion.

So $\{B(t), t > T_a\} = \{a + B(t) - a, t > T_a\}$

$$\sim \{a + a - B(t), t > T_a\} = \{2a - B(t), t > T_a\}.$$

Brownian Motion Absorbed at a Value

Let $\{B(t)\}$ be a Brownian Motion.
For $a > 0$, a Brownian Motion absorbed at a value a is defined as

$$B_a(t) = \begin{cases} B(t) & \text{if } \max_{0 \le s \le t} B(s) < a \\ a & \text{if } \max_{0 \le s \le t} B(s) \ge a \end{cases}$$

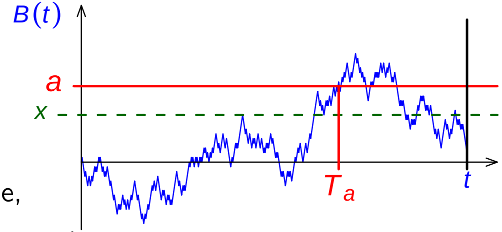
What is the distribution of $B_a(t)$? For $x < a$,

$$\begin{aligned} \mathrm{P}(B_a(t) \le x) &= \mathrm{P}\Big(B(t) \le x, \max_{0 \le s \le t} B(s) < a\Big) \\ &= \mathrm{P}(B(t) \le x) - \mathrm{P}\Big(B(t) \le x, \max_{0 \le s \le t} B(s) \ge a\Big) \\ &= \mathrm{P}(B(t) \le x) - \mathrm{P}(B(t) \le x, T_a \le t) \end{aligned}$$

where the last equality comes from the fact

$$\Big\{\max_{0 \le s \le t} B(s) \ge a\Big\} \Leftrightarrow \{T_a \le t\}.$$

Brownian Motion Absorbed at a Value



By the Reflection principle,

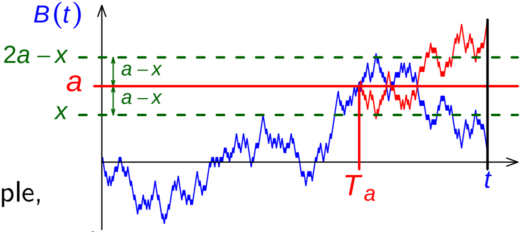
$$
\begin{aligned}
&\mathrm{P}(B(t) \leq x, T_a \leq t) \\
&= \mathrm{P}(B(t) \geq 2a - x, T_a \leq t) = \mathrm{P}(B(t) \geq 2a - x)
\end{aligned}
$$

since $x \leq a$, $B(t) \geq 2a - x > a$ implies $T_a \leq t$.

In summary, the CDF of $B_a(t)$ is

$$
\begin{aligned}
\mathrm{P}(B_a(t) \leq x) &= \mathrm{P}\left(B(t) \leq x, \max_{0 \leq s \leq t} B(s) < a\right) \\
&= \mathrm{P}(B(t) \leq x) - \mathrm{P}(B(t) \geq 2a - x) \\
&= \Phi\left(\frac{x}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2a - x}{\sqrt{t}}\right)
\end{aligned}
$$

Brownian Motion Absorbed at a Value



By the Reflection principle,

$$\begin{aligned}&\mathrm{P}(B(t) \leq x,\, T_a \leq t)\\ &= \mathrm{P}(B(t) \geq 2a - x,\, T_a \leq t) = \mathrm{P}(B(t) \geq 2a - x)\end{aligned}$$

since $x \leq a$, $B(t) \geq 2a - x > a$ implies $T_a \leq t$.

In summary, the CDF of $B_a(t)$ is

$$\begin{aligned}\mathrm{P}(B_a(t) \leq x) &= \mathrm{P}\left(B(t) \leq x,\, \max_{0\leq s\leq t} B(s) < a\right)\\ &= \mathrm{P}(B(t) \leq x) - \mathrm{P}(B(t) \geq 2a - x)\\ &= \Phi\left(\frac{x}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2a-x}{\sqrt{t}}\right)\end{aligned}$$

More on the Reflection Principle

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Let's try to find the joint distribution of

$$W(t) = \max_{0 \leq s \leq t} B(s) \quad \text{and} \quad Y(t) = W(t) - B(t)$$

By the Reflection Principle,

$$\mathrm{P}(W(t) \geq w,\ B(t) \leq x) = \mathrm{P}(T_w \leq t,\ B(t) \leq x) = \mathrm{P}(B(t) \geq 2w - x)$$

The joint CDF of $W(t)$ and $B(t)$ is hence,

$$\begin{aligned}
\mathrm{P}(W(t) \leq w,\ B(t) \leq x) &= \mathrm{P}(B(t) \leq x) - \mathrm{P}(W(t) \geq w,\ B(t) \leq x) \\
&= \mathrm{P}(B(t) \leq x) - \mathrm{P}(B(t) \geq 2w - x) \\
&= \Phi\Big(\frac{x}{\sqrt{t}}\Big) - \Big[1 - \Phi\Big(\frac{2w - x}{\sqrt{t}}\Big)\Big].
\end{aligned}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$ is the CDF of $N(0,1)$.

Let $\phi(x) = \frac{d}{dx}\Phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ be the density of $N(0,1)$,
Observe that the derivative of $\phi(x)$ is

$$\phi'(x) = \frac{d}{dx}\phi(x) = \frac{-x}{\sqrt{2\pi}} e^{-x^2/2} = -x\phi(x).$$

Take the derivative of the joint CDF of $W(t)$ and $B(t)$ on the previous slide with respect to w and x we get the joint density of $W(t)$ and $B(t)$ below

$$\begin{aligned}
f(w,x) &= \frac{d}{dx}\frac{d}{dw}\left\{\Phi\Big(\frac{x}{\sqrt{t}}\Big) - 1 + \Phi\Big(\frac{2w-x}{\sqrt{t}}\Big)\Big]\right\} \\
&= \frac{d}{dx}\Big[0 + \frac{2}{\sqrt{t}}\phi\Big(\frac{2w-x}{\sqrt{t}}\Big)\Big] \qquad \text{(since } \frac{d}{dw}\left(\Phi\Big(\frac{x}{\sqrt{t}}\Big) - 1\right) = 0) \\
&= \frac{2w-x}{t}\frac{2}{\sqrt{t}}\phi\Big(\frac{2w-x}{\sqrt{t}}\Big) \qquad \text{(since } \phi'(x) = -x\phi(x)) \\
&= \sqrt{\frac{2}{\pi t^3}}(2w-x)\exp\Big(-\frac{(2w-x)^2}{2t}\Big),\ w \geq 0,\ x \leq w
\end{aligned}$$

Thus the joint density of $W(t)$ and $B(t)$ is

$$f(w,x) = \sqrt{\frac{2}{\pi t^3}}(2w-x)\exp\Big(-\frac{(2w-x)^2}{2t}\Big),\ w \geq 0,\ x \leq w$$

By a change of variable of $W(t)$, $Y(t) = W(t) - B(t)$, we can find the desired joint density of $W(t)$, and $Y(t)$

$$\begin{aligned} g(w,y) &= f(w, w-y) \\ &= \sqrt{\frac{2}{\pi t^3}}(w+y)\exp\Big(-\frac{(w+y)^2}{2t}\Big),\ w \geq 0,\ y \geq 0 \end{aligned}$$

Note that the density is symmetric in w and y.
Thus $Y(t)$ has the same marginal distribution as $W(t)$, which is also same as $|B(t)|$.