STAT253/317 Lecture 11

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Superposition

The sum of two independent Poisson processes with respective rates λ_1 and λ_2, called the **superposition** of the processes, is again a Poisson process but with rate $\lambda_1 + \lambda_2$.

The proof is straight forward from Definition 5.3 and hence omitted.

Remark: By repeated application of the above arguments we can see that the superposition of k independent Poisson processes with rates $\lambda_1, \cdots, \lambda_k$ is again a Poisson process with rate $\lambda_1 + \cdots + \lambda_k$.

Why Are Poisson Processes Commonly Used?

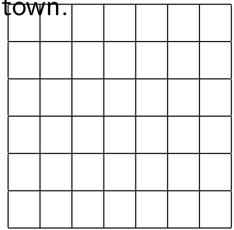
There is a useful result in probability theory which says that:

> *if we take N independent counting processes and sum them up, then the resulting superposition process is approximately a Poisson process.*

Here

- N must be "large enough" and
- the rates of the individual processes must be "small" relative to N
- but the individual processes that go into the superposition can otherwise be arbitrary.

E.g., $N(t) = \#$ of crimes by time t in a certain town.



Thinning

Consider a Poisson process $\{N(t) : t \geq 0\}$ with rate λ.
At each arrival of events, it is classified as a

$$\begin{cases} \text{Type 1 event with probability} & p & \text{or} \\ \text{Type 2 event with probability} & 1-p, \end{cases}$$

independently of all other events. Let

$$N_i(t) = \# \text{ of type } i \text{ events occurred during } [0,t], \; i = 1, 2.$$

Note that $N(t) = N_1(t) + N_2(t)$.

Proposition 5.2

$\{N_1(t), t \geq 0\}$ and $\{N_2(t), t \geq 0\}$ are both Poisson processes having respective rates λp and $\lambda(1-p)$.

Furthermore, the two processes are independent.

Proof of Proposition 5.2

First observe that given $N(t) = n + m$,

$$N_1(t) \sim \mathit{Binomial}(n + m, p). \qquad \text{(why?)}$$

Thus $\mathrm{P}(N_1(t) = n, N_2(t) = m)$

$$= \mathrm{P}(N_1(t) = n, N_2(t) = m | N(t) = n + m)\mathrm{P}(N(t) = n + m)$$

$$= \binom{n+m}{n} p^n (1-p)^m e^{-\lambda t} \frac{(\lambda t)^{n+m}}{(n+m)!}$$

$$= e^{-\lambda t p} \frac{(\lambda p t)^n}{n!} e^{-\lambda t(1-p)} \frac{(\lambda(1-p)t)^m}{m!}$$

$$= \mathrm{P}(N_1(t) = n)\mathrm{P}(N_2(t) = m).$$

This proves the independence of $N_1(t)$ and $N_2(t)$ and that

$$N_1(t) \sim \mathit{Poisson}(\lambda p t), \quad N_2(t) \sim \mathit{Poisson}(\lambda(1-p)t).$$

Both $\{N_1(t)\}$ and $\{N_2(t)\}$ inherit the stationary and independent increment properties from $\{N(t)\}$, and hence are both Poisson processes.

Some "Converse" of Thinning & Superposition

Consider two indep. Poisson processes $\{N_A(t)\}$ and $\{N_B(t)\}$ w/ respective rates λ_A and λ_B. Let

$$S_n^A = \text{arrival time of the } n\text{th } A \text{ event}$$
$$S_m^B = \text{arrival time of the } m\text{th } B \text{ event}$$

Find $\mathrm{P}(S_n^A < S_m^B)$.

Approach 1:
Observer that $S_n^A \sim Gamma(n, \lambda_A)$, $S_m^B \sim Gamma(m, \lambda_B)$ and they are independent. Thus

$$\mathrm{P}(S_n^A < S_m^B) = \int_{x<y} \lambda_A e^{-\lambda_A x} \frac{(\lambda_A x)^{n-1}}{(n-1)!} \lambda_B e^{-\lambda_B y} \frac{(\lambda_B y)^{m-1}}{(m-1)!} dx dy$$

Some "Converse" of Thinning & Superposition (Cont'd)

Let $N(t) = N_A(t) + N_B(t)$ be the superposition of the two processes. Let

$$I_i = \begin{cases} 1 & \text{if the } i\text{th event in the superpositon process is an } A \text{ event} \\ 0 & \text{otherwise} \end{cases}.$$

The I_i, $i = 1, 2, \dots$ are i.i.d. Bernoulli(p), where $p = \dfrac{\lambda_A}{\lambda_A + \lambda_B}$. mth tail

Approach 2: ~~HHHTTHTTTLTT~~ T

$$\mathrm{P}(S_n^A < S_1^B) = \mathrm{P}(\text{the first } n \text{ events are all } A \text{ events}) = \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^n$$

$$\begin{aligned} \mathrm{P}(S_n^A < S_m^B) &= \mathrm{P}(\text{at least } n\ A \text{ events occur before } m\ B \text{ events}) \\ &= \mathrm{P}(\text{at least } n \text{ heads before } m \text{ tails}) \\ &= \mathrm{P}(\text{at least } n \text{ heads in the first } n+m-1 \text{ tosses}) \\ &= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_A}{\lambda_A + \lambda_B}\right)^k \left(\frac{\lambda_B}{\lambda_A + \lambda_B}\right)^{n+m-1-k} \end{aligned}$$

Proposition 5.3 (Generalization of Proposition 5.2)

Consider a Poisson process with rate λ. If an event occurs at time t will be classified as a type i event with probability $p_i(t)$, $i = 1, \ldots, k$, $\sum_i p_i(t) = 1$, for all t, independently of all other events. then

$$N_i(t) = \text{number of type } i \text{ events occurring in } [0, t], \; i = 1, \ldots, k.$$

Note $N(t) = \sum_{i=1}^k N_i(t)$. Then $N_i(t)$, $i = 1, \ldots, k$ are independent Poisson random variables with means $\lambda \int_0^t p_i(s) ps$.

Remark: Note $\{N_i(t), t \geq 0\}$ are NOT Poisson processes.

Example

- Policyholders of a certain insurance company have accidents occurring according to a Poisson process with rate λ.
- The amount of time T from when the accident occurs until a claim is made has distribution $G(t) = \mathrm{P}(T \le t)$.
- Let $N_c(t)$ be the number of claims made by time t.

Find the distribution of $N_c(t)$.

Solution. Suppose an accident occurred at time s. It is claimed by time t if $s + T \le t$, i.e., with probability

P(s + T <= t)

$$p(s) = \mathrm{P}(T \le t - s) = G(t - s).$$

By Proposition 5.3, $N_c(t)$ has a Poisson distribution with mean

$$\lambda \int_0^t p(s)ps = \lambda \int_0^t G(t-s)ds = \lambda \int_0^t G(s)ds$$

5.4.1 Nonhomogeneous Poisson Process

Definition 5.4a. A nonhomogeneous (a.k.a. non-stationary) Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$.

(ii) having independent increments.

(iii) $\mathrm{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$.

(iv) $\mathrm{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Definition 5.4b. A nonhomogeneous Poisson process with intensity function $\lambda(t) \geq 0$ is a counting process $\{N(t), t \geq 0\}$ satisfying

(i) $N(0) = 0$,

(ii) for $s, t \geq 0$, $N(t+s) - N(s)$ is independent of $N(s)$ (independent increment)

(iii) For $s, t \geq 0$, $N(t+s) - N(s) \sim Poisson(m(t+s) - m(s))$, where $m(t) = \int_0^t \lambda(u)du$

The two definitions are equivalent.

The Interarrival Times of a Nonhomogeneous Poisson Process Are NOT Independent

A nonhomogeneous Poisson process **has independent increment** but its **interarrival times** between events are

- neither independent
- nor identically distributed.

Proof. Homework.

Proposition 5.4

Let $\{N_1(t), t \geq 0\}$, and $\{N_2(t), t \geq 0\}$ be two independent nonhomogeneous Poisson process with respective intensity functions $\lambda_1(t)$ and $\lambda_2(t)$, and let $N(t) = N_1(t) + N_2(t)$. Then

(a) $\{N(t), t \geq 0\}$ is a nonhomogeneous Poisson process with intensity function $\lambda_1(t) + \lambda_2(t)$.

(b) Given that an event of the $\{N(t), t \geq 0\}$ process occurs at time t then, independent of what occurred prior to t , the event at t was from the $\{N_1(t)\}$ process with probability

$$\frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}.$$

5.4.2 Compound Poisson Processes

Definition. Let $\{N(t)\}$ be a (homogeneous) Poisson process with rate λ and Y_1, $Y_2, \ldots$ are i.i.d random variables independent of $\{N(t)\}$. The process

$$X(t) = \sum_{i=1}^{N(t)} Y_i$$

is called a *compound Poisson process*, in which $X(t)$ is defined as 0 if $N(t) = 0$.

A compound Poisson process has

- **independent increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ is independent of $X(s) = \sum_{i=1}^{N(s)} Y_i$, and
- **stationary increment**, since $X(t+s) - X(s) = \sum_{i=1}^{N(t+s)-N(s)} Y_{i+N(s)}$ has the same distribution as $X(t) = \sum_{i=1}^{N(t)} Y_i$

The Mean of a Compound Poisson Process

Suppose $\mathbb{E}[Y_i] = \mu_Y$, $\mathrm{Var}(Y_i) = \sigma_Y^2$. Note that $\mathbb{E}[N(t)] = \lambda t$.

$$\begin{aligned}
\mathbb{E}[X(t)|N(t)] &= \sum_{i=1}^{N(t)} \mathbb{E}[Y_i|N(t)] \\
&= \sum_{i=1}^{N(t)} \mathbb{E}[Y_i] \quad \text{(since } Y_i\text{'s are indep. of } N(t)\text{)} \\
&= N(t)\mu_Y
\end{aligned}$$

Thus

$$\mathbb{E}[X(t)] = \mathbb{E}[\mathbb{E}[X(t)|N(t)]] = \mathbb{E}[N(t)]\mu_Y = \lambda t \mu_Y$$

Variance of a Compound Poisson Process (Cont'd)

Similarly, using that $\mathbb{E}[N(t)] = \mathrm{Var}(N(t)) = \lambda t$, we have

$$
\begin{aligned}
\mathrm{Var}[X(t)|N(t)] &= \mathrm{Var}\left(\sum_{i=1}^{N(t)} Y_i \middle| N(t)\right) \\
&= \sum_{i=1}^{N(t)} \mathrm{Var}(Y_i|N(t)) \\
&= \sum_{i=1}^{N(t)} \mathrm{Var}(Y_i) \quad \text{(since } Y_i\text{'s are indep. of } N(t)) \\
&= N(t)\sigma_Y^2 \\
\mathbb{E}[\mathrm{Var}(X(t)|N(t))] &= \mathbb{E}[N(t)\sigma_Y^2] = \lambda t \sigma_Y^2 \\
\mathrm{Var}(\mathbb{E}[X(t)|N(t)]) &= \mathrm{Var}(N(t)\mu_Y) = \mathrm{Var}(N(t))\mu_Y^2 = \lambda t \mu_Y^2
\end{aligned}
$$

Thus

$$
\begin{aligned}
\mathrm{Var}(X(t)) &= \mathbb{E}[\mathrm{Var}[X(t)|N(t)]] + \mathrm{Var}(\mathbb{E}[X(t)|N(t)]) \\
&= \lambda t(\sigma_Y^2 + \mu_Y^2) = \lambda t \mathbb{E}[Y_i^2]
\end{aligned}
$$

CLT of a Compound Poisson Process

As $t \to \infty$, the distribution of

$$\frac{X(t) - \mathbb{E}[X(t)]}{\sqrt{\mathrm{Var}(X(t))}} = \frac{X(t) - \lambda t \mu_Y}{\sqrt{\lambda t(\sigma_Y^2 + \mu_Y^2)}}$$

converges to a standard normal distribution $N(0, 1)$.

5.4.3 Conditional Poisson Processes

Definition. A *conditional* (or *mixed*) *Poisson process* $\{N(t), t \geq 0\}$ is a counting process satisfying

(i) $N(0) = 0$,

(ii) having stationary increment, and

(iii) there is a random variable $\Lambda > 0$ with probability density $g(\lambda)$, such that given $\Lambda = \lambda$,

$$N(t+s) - N(s) \sim Poisson(\lambda t),$$

i.e.,

$$\mathrm{P}(N(t+s) - N(s) = k) = \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda, \; k = 0, 1, \ldots$$

Remark: In general, a conditional Poisson process does NOT have independent increment.

$$
\begin{aligned}
&\mathrm{P}(N(s) = j, N(t+s) - N(s) = k) \\
&= \int_0^\infty e^{-\lambda s} \frac{(\lambda s)^j}{j!} e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda \\
&\neq \left(\int_0^\infty e^{-\lambda s} \frac{(\lambda s)^j}{j!} g(\lambda) d\lambda \right) \left(\int_0^\infty e^{-\lambda t} \frac{(\lambda t)^k}{k!} g(\lambda) d\lambda \right) \\
&= \mathrm{P}(N(s) = j) \mathrm{P}(N(t+s) - N(s) = k)
\end{aligned}
$$