# STAT253/317 Lecture 10

Yibi Huang

5.3 The Poisson Processes

# Properties of Poisson Processes

Outline:

- Interarrival times of events are i.i.d Exponential with rate $\lambda$
- Conditional Distribution of the Arrival Times
- Superposition & Thinning ......................(Lecture 11)
- "Converse" of Superposition & Thinning .......(Lecture 11)

# Arrival & Interarrival Times of Poisson Processes

Let

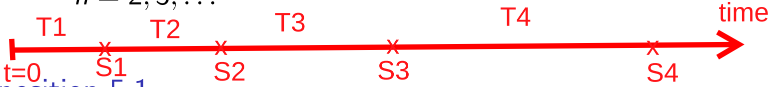
$$S_n = \text{Arrival time of the } n\text{-th event}, n = 1, 2, \ldots$$

$$T_1 = S_1 = \text{Time until the 1st event occurs}$$

$$T_n = S_n - S_{n-1}$$

$$= \text{time elapsed between the } (n-1)\text{st and } n\text{-th event,}$$

$$n = 2, 3, \ldots$$




## Proposition 5.1

The interarrival times $T_1, T_2, \ldots, T_k, \ldots,$ are i.i.d $\sim Exp(\lambda)$.

Consequently, as the distribution of the sum of $n$ i.i.d Exp($\lambda$) is $Gamma(n, \lambda)$, the arrival time of the $n$th event is

$$S_n = \sum_{i=1}^{n} T_i \sim Gamma(n, \lambda)$$

# Proof of Proposition 5.1

$$
\begin{aligned}
& \mathrm{P}(T_{n+1} > t \mid T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
= {} & \mathrm{P}(0 \text{ event in } (s_n, s_n + t] \mid T_1 = t_1, T_2 = t_2, \ldots, T_n = t_n) \\
& \qquad\qquad\qquad\qquad (\text{where } s_n = t_1 + t_2 + \cdots + t_n) \\
= {} & \mathrm{P}(0 \text{ event in } (s_n, s_n + t]) \quad (\text{by indep increment}) \\
= {} & \mathrm{P}(N(s_n + t) - N(s_n) = 0) \\
= {} & e^{-\lambda t} \quad (\text{stationary increment})
\end{aligned}
$$

This shows that $T_{n+1}$ is $\sim Exp(\lambda)$, and is independent of $T_1, T_2, \ldots, T_n$.

# Definition 3 of the Poisson Process

A continuous-time stochastic process $\{N(t), t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ if

(i) $N(0) = 0$,

(ii) $N(t)$ counts the number of events that have occurred up to time t (i.e. it is a counting process).

(iii) The times between events are independent and identically distributed with an Exponential($\lambda$) distribution.

We have seen how Definition 5.1 implies (i), (ii) and (iii) in Definition 3. The proof of the converse is omitted.

# 5.3.5. Conditional Distribution of the Arrival Times

## Uniform Distribution of Arrivals

Given $N(t) = 1$, then $T_1$, the arrival time of the first event $\sim Unif(0, t)$

*Proof.*


$$\mathrm{P}(T_1 \leq s | N(t) = 1) = \frac{\mathrm{P}(T_1 \leq s, N(t) = 1)}{\mathrm{P}(N(t) = 1)}$$

P(N(s)=1)P(N(t)-N(s)=0)

$$= \frac{\mathrm{P}(1 \text{ event in } (0, s], \text{ no events in } (s, t])}{\mathrm{P}(N(t) = 1)}$$

$$= \frac{(\lambda s e^{-\lambda s})(e^{-\lambda(t-s)})}{\lambda t e^{-\lambda t}} = \frac{s}{t}, s < t.$$

which is the CDF of the Uniform (0,t) distribution.

# Theorem 5.2

Given $N(t) = n$, the n events are equally likely to occur anywhere between 0 and t, all Uniform(0,t).

$$(S_1, S_2, \ldots, S_n) \sim (U_{(1)}, U_{(2)}, \ldots, U_{(n)})$$

where $(U_{(1)}, \ldots, U_{(k)})$ are the order statistics of $(U_1, \ldots, U_n) \sim$ i.i.d Uniform $(0, t)$, i.e., the joint conditional density of $S_1, S_2, \ldots$, $S_n$ is S1 = first event = min(U1, U2,..., Un)

$$f(s_1, s_2, \ldots, s_n | N(t) = n) = n!/t^n,\ 0 < s_1 < s_2 < \ldots < s_n < t$$

*Proof.* The event that $S_1 = s_1$, $S_2 = s_2, \ldots$, $S_n = s_n$, $N(t) = n$ is equivalent to the event $T_1 = s_1$, $T_2 = s_2 - s_1, \ldots, T_n = s_n - s_{n-1}$, $T_{n+1} > t - s_n$. Hence, by Proposition 5.1, we have the conditional joint density of $S_1, \ldots, S_n$ given $N(t) = n$ as follows:

$$\begin{aligned} f(s_1, \ldots, s_n | n) &= \frac{f(s_1, \ldots, s_n, n)}{\mathrm{P}(N(t) = n)} \\ &= \frac{\lambda e^{-\lambda s_1} \lambda e^{-\lambda(s_2 - s_1)} \ldots \lambda e^{-\lambda(s_n - s_{n-1})} e^{-\lambda(t - s_n)}}{e^{-\lambda t}(\lambda t)^n / n!} \\ &= n! t^{-n}, \quad 0 < s_1 < \ldots < s_n < t \end{aligned}$$

**Example 5.21**. Insurance claims comes according to a Poisson process $\{N(t)\}$ with rate $\lambda$. Let

- $S_i$ = the time of the $i$th claims
- $C_i$ = amount of the $i$th claims, i.i.d with mean $\mu$, indep. of $\{N(t)\}$

Then the total discounted cost by time $t$ at discount rate $\alpha$ is given by

$$D(t) = \sum_{i=1}^{N(t)} C_i e^{-\alpha S_i}.$$

Then

$$\begin{aligned}\mathbb{E}[D(t)|N(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha S_i} \middle| N(t)\right] \overset{(Thm\ 5.2)}{=} \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_{(i)}}\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{N(t)} C_i e^{-\alpha U_i}\right] = \sum_{i=1}^{N(t)} \mathbb{E}[C_i]\mathbb{E}\left[e^{-\alpha U_i}\right] \\ &= N(t)\mu \int_0^t \frac{1}{t} e^{-\alpha x} dx = N(t)\frac{\mu}{\alpha t}(1 - e^{-\alpha t})\end{aligned}$$

Thus $\mathbb{E}[D(t)] = \mathbb{E}[N(t)]\frac{\mu}{\alpha t}(1 - e^{\alpha t}) = \frac{\lambda\mu}{\alpha}(1 - e^{-\alpha t})$