

STAT253/317 Winter 2014 Lecture 9

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Chapter 5 Poisson Processes

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5.2 Exponential Distribution

Let X be of exponential distribution with rate λ : $X \sim \text{Exp}(\lambda)$.

- ▶ Density: $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$
- ▶ CDF: $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$
- ▶ Moment generating function

$$M_X(t) = \mathbb{E}[e^{tX}] = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

- ▶ $\mathbb{E}(X) = 1/\lambda$, $\text{Var}(X) = 1/\lambda^2$
- ▶ If X_1, \dots, X_n are i.i.d $\text{Exp}(\lambda)$, then $S_n = X_1 + \dots + X_n \sim \text{Gamma}(n, \lambda)$, with density

$$f_{S_n}(x) = \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}$$

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The Exponential Distribution is Memoryless (*****)

$$P(X > t + s | X > t) = P(X > s)$$

Proof.

$$\begin{aligned} P(X > t + s | X > t) &= \frac{P(X > t + s \text{ and } X > t)}{P(X > t)} \\ &= \frac{P(X > t + s)}{P(X > t)} \\ &= \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = e^{-\lambda s} = P(X > s) \end{aligned}$$

Implication. If the lifetime of batteries has an Exponential distribution, then a used battery is as good as a new one, as long as it's not dead!

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Another Important Property of the Exponential

If X_1, \dots, X_n are independent, $X_i \sim \text{Exp}(\lambda_i)$ for $i = 1, \dots, n$ then

- (i) $\min(X_1, \dots, X_n) \sim \text{Exp}(\lambda_1 + \dots + \lambda_n)$, and
- (ii) $P(X_j = \min(X_1, \dots, X_n)) = \frac{\lambda_j}{\lambda_1 + \dots + \lambda_n}$

Proof of (i)

$$\begin{aligned} P(\min(X_1, \dots, X_n) > t) &= P(X_1 > t, \dots, X_n > t) \\ &= P(X_1 > t) \dots P(X_n > t) = e^{-\lambda_1 t} \dots e^{-\lambda_n t} \\ &= e^{-(\lambda_1 + \dots + \lambda_n)t}. \end{aligned}$$

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Proof of (ii)

$$\begin{aligned} &P(X_j = \min(X_1, \dots, X_n)) \\ &= P(X_j < X_i \text{ for } i = 1, \dots, n, i \neq j) \\ &= \int_0^\infty P(X_j < X_i \text{ for } i \neq j | X_j = t) \lambda_j e^{-\lambda_j t} dt \\ &= \int_0^\infty P(t < X_i \text{ for } i \neq j) \lambda_j e^{-\lambda_j t} dt \\ &= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} P(X_i > t) dt \\ &= \int_0^\infty \lambda_j e^{-\lambda_j t} \prod_{i \neq j} e^{-\lambda_i t} dt \\ &= \lambda_j \int_0^\infty e^{-(\lambda_1 + \dots + \lambda_n)t} dt \\ &= \frac{\lambda_j}{\lambda_1 + \dots + \lambda_n} \end{aligned}$$

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Example 5.8: Post Office (p.305)

- ▶ A post office has two clerks.
- ▶ Service times for clerk $i \sim \text{Exp}(\lambda_i)$, $i = 1, 2$
- ▶ When you arrive, both clerks are busy but no one else waiting. You will enter service when either clerk becomes free.
- ▶ Find $\mathbb{E}[T]$, where T = the amount of time you spend in the post office.

Solution. Let R_i = remaining service time of the customer with clerk i , $i = 1, 2$.

- ▶ Note R_i 's are indep. $\sim \text{Exp}(\lambda_i)$, $i = 1, 2$ by the memoryless property
- ▶ Observe $T = \min(R_1, R_2) + S$ where S is your service time
- ▶ Using the property of exponential distributions,

$$\min(R_1, R_2) \sim \text{Exp}(\lambda_1 + \lambda_2) \Rightarrow \mathbb{E}[\min(R_1, R_2)] = \frac{1}{\lambda_1 + \lambda_2}$$

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Example 5.8: Post Office (p.305, Cont'd)

As for your service time S , observe that

$$S \sim \begin{cases} \text{Exp}(\lambda_1) & \text{if } R_1 < R_2 \\ \text{Exp}(\lambda_2) & \text{if } R_2 < R_1 \end{cases} \Rightarrow \begin{cases} \mathbb{E}[S|R_1 < R_2] = 1/\lambda_1 \\ \mathbb{E}[S|R_2 < R_1] = 1/\lambda_2 \end{cases}$$

Recall that $P(R_1 < R_2) = \lambda_1/(\lambda_1 + \lambda_2)$ So

$$\begin{aligned} \mathbb{E}[S] &= \mathbb{E}[S|R_1 < R_2]P(R_1 < R_2) + \mathbb{E}[S|R_2 < R_1]P(R_2 < R_1) \\ &= \frac{1}{\lambda_1} \times \frac{\lambda_1}{\lambda_1 + \lambda_2} + \frac{1}{\lambda_2} \times \frac{\lambda_2}{\lambda_1 + \lambda_2} = \frac{2}{\lambda_1 + \lambda_2} \end{aligned}$$

Hence the expected amount of time you spend in the post office is

$$\begin{aligned} \mathbb{E}[T] &= \mathbb{E}[\min(R_1, R_2)] + \mathbb{E}[S] \\ &= \frac{1}{\lambda_1 + \lambda_2} + \frac{2}{\lambda_1 + \lambda_2} = \frac{3}{\lambda_1 + \lambda_2}. \end{aligned}$$

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Definition.

A process $\{X(t), t \geq 0\}$ is said to have *stationary increments* if for any $t > s$, the distribution of $X(t) - X(s)$ depends on s and t only through the difference $t - s$, for all $s < t$.

That is, $X(t + a) - X(s + a)$ has the same distribution as $X(t) - X(s)$ for any constant a .

Definition.

A process $\{X(t), t \geq 0\}$ is said to have *independent increments* if for any $s_1 < t_1 \leq s_2 < t_2 \leq \dots \leq s_k < t_k$, the random variable $X(t_1) - X(s_1), X(t_2) - X(s_2), \dots, X(t_k) - X(s_k)$ are independent, i.e. the numbers of events that occur in **disjoint** time intervals are **independent**.

Example. Simple random walk $\{X_n, n \geq 0\}$ is a process with independent and stationary increment, since $X_n = \sum_{k=0}^n \xi_k$ where ξ_k 's are i.i.d with $P(\xi_k = 1) = p$ and $P(\xi_k = -1) = 1 - p$.

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Definition 5.3 of Poisson Processes

The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process having rate $\lambda, \lambda > 0$, if

- (i) $N(0) = 0$.
- (ii) The process has stationary and independent increments.
- (iii) $P(N(h) = 1) = \lambda h + o(h)$.
- (iv) $P(N(h) \geq 2) = o(h)$.

Theorem 5.1 Definitions 5.1 and 5.3 are equivalent.

[Proof of Definitions 5.1 \Rightarrow Definition 5.3]

From Definitions 5.1, $N(h) \sim \text{Poi}(h)$. Thus

$$\begin{aligned} P(N(h) = 1) &= \lambda h e^{-\lambda h} = \lambda h + o(h) \\ P(N(h) \geq 2) &= 1 - P(N(h) = 0) - P(N(h) = 1) \\ &= 1 - e^{-\lambda h} - \lambda h e^{-\lambda h} = o(h) \end{aligned}$$

Proof of Definitions 5.3 \Rightarrow Definition 5.1: See p.315 in [IPM10e]

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5.3.1. Counting Processes

A counting process $\{N(t)\}$ is a cumulative count of number of events happened up to time t .

Definition.

A stochastic processes $\{N(t), t \geq 0\}$ is a *counting process* satisfying

- (i) $N(t) = 0, 1, \dots$ (integer valued),
- (ii) If $s < t$, then $N(s) \leq N(t)$.
- (iii) For $s < t$, $N(t) - N(s) =$ number of events that occur in the interval $(s, t]$.

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Definition 5.1 of Poisson Processes

A Poisson process with rate $\lambda > 0$ $\{N(t), t \geq 0\}$ is a counting process satisfying

- (i) $N(0) = 0$,
- (ii) For $s < t$, $N(t) - N(s)$ is independent of $N(s)$ (independent increment)
- (iii) For $s < t$, $N(t) - N(s) \sim \text{Poi}(\lambda(t - s))$, i.e.,

$$P(N(t) - N(s) = k) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^k}{k!}$$

Remark: In (iii), the distribution of $N(t) - N(s)$ depends on $t - s$ only, not s , which implies $N(t)$ has stationary increment.

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