Generating Functions

A generating function of a non-negative integer-valued random variable \( T \) is the expected value of \( s^T \):

\[
G(s) = E[s^T] = \sum_{k=0}^{\infty} s^k P(T = k),
\]

in which \( s^T \) is defined as 0 if \( T = \infty \).

- \( G(s) \) is a power series converging absolutely for all \(-1 < s < 1\).
- \( G(1) = P(T < \infty) \begin{cases} = 1 & \text{if } T \text{ is finite w/ prob. 1} \\ < 1 & \text{otherwise} \end{cases} \)

Knowing \( G(s) \) is equivalent to knowing \( P(T = k) \) for all \( k = 0, 1, 2, \ldots \)

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space \( = \{0, 1, 2, 3, \ldots\} \) with \( P(0, 1) = P(i, i+1) = p \) and \( P(i, i-1) = 1 - p = q \) for \( i = 1, 2, 3 \ldots \)
- Only one class, irreducible
- For \( i < j \), define

\[
N_{ij} = \min\{m > 0 : X_m = j | X_0 = i\}
\]

- Time to reach state \( j \) starting in state \( i \)
- Observe that \( N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n} \)

By the Markov property, \( N_{01}, N_{12}, \ldots, N_{n-1,n} \) are independent.

- Given \( X_0 = i \),

\[
N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i + 1 \\ 1 + N_{i-1,i}^* + N_{i,i+1}^* & \text{if } X_1 = i - 1 \end{cases}
\]

where \( N_{i,j}^* \) are independent.

Similarly,

\[
G_2(s) = \left. \frac{ps}{1 - qst} \right|_{s=1} = \frac{ps(1-q^2)}{1-q(1+p)s^2} = \frac{ps}{1-q(1+p)s^2} - \frac{pq^3}{1-q(1+p)s^2} = \sum_{k=0}^{\infty} p^k \left( \frac{1}{1+p} \right)^k s^k \sum_{k=0}^{\infty} q(1+p)s^{2k+1} - \sum_{k=0}^{\infty} pq^{k+1}(1+p)s^{2k+3} = \sum_{k=0}^{\infty} p^k q^k \left( \frac{1}{1+p} \right)^k s^{2k+1}
\]

So

\[
P(N_{23} = n) = \begin{cases} p & \text{if } n = 1 \\ p^2 q \left( \frac{1}{1+p} \right)^{k-1} & \text{if } n = 2k + 1 \text{ for } k = 1, 2, \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}
\]
Mean of $N_{i,j+1}$
Recall that $G'_i(1) = E(N_{i,j+1})$. Let $m_i = E(N_{i,j+1}) = G'_i(1)$.

\[
G'_i(s) = \frac{p(1 - qG_{i-1}(s)) + ps(qG_{i-1}(s) + qG'_{i-1}(s))}{s(1 - qG_{i-1}(s))^2}
\]

\[
= \frac{p + psG'_{i-1}(s)}{(1 - qG_{i-1}(s))^2}
\]

Since $N_{i,j+1} < \infty$, $G_i(1) = 1$ for all $i = 0, 1, \ldots n - 1$. We have

\[
m_i = G'_i(1) = \frac{1 + qG'_{i-1}(1)}{1 - qG'_{i-1}(1)} = \frac{q}{1 - q}
\]

Since $m_0 = 1$, which implies $m_i = 1$.

\[
m_i = \begin{cases} 
\frac{1 - (q/p)}{p} + \left(2 \frac{q}{p}\right)^i & \text{if } p \neq 0.5 \\
2i + 1 & \text{if } p = 0.5
\end{cases}
\]

Example: Symmetric Random Walk on $(-\infty, \infty)$
State space $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

\[
P_{i,i+1} = P_{i,i-1} = 1/2, \quad \text{for all integer } i
\]

- $1$ classes, recurrent, null-recurrent or positive-recurrent?
- For $i, j$, define $N_j = \min\{m > 0 : X_m = j|X_0 = i\}$.
- Note $N_0 = 1 + N_{10}$
- Given $X_0 = 1$

\[
N_{10} = \begin{cases} 
1 & \text{if } X_1 = 0 \\
1 + N_{11} + N'_{10} & \text{if } X_1 = 2
\end{cases}
\]

Note $N_{21}$ and $N'_{10}$ are independent and have the same distribution as $N_{10}$ (Why?)

\[\text{Lecture 8 - 11}\]

Generating Function of $N_{10}$
Let $G(s)$ be the generating function of $N_{10}$. From (3), we know that

\[
G(s) = \frac{1}{2} + \frac{1}{2}E[s^{1+N_{01}+N'_{01}}] = \frac{1}{2} s + \frac{1}{2} s G(s)^2
\]

which is a quadratic equation in $G(s)$. The two roots are

\[
G(s) = \frac{1 \pm \sqrt{1 - s^2}}{s}
\]

Since $G(s)$ must lie between 0 and 1 when $0 < s < 1$. So

\[
G(s) = \frac{1 - \sqrt{1 - s^2}}{s}
\]

Note that

\[
G'(s) = \frac{1}{\sqrt{1 - s^2} + 1 - s^2}, \quad E[N_{10}] = \lim_{s \to 1^-} G'(s) = \infty
\]

which implies $E[N_{00}] = 1 + E[N_{10}] = \infty$.

$\Rightarrow$ Symmetric random walk is null recurrent.

\[\text{Lecture 8 - 9}\]

From all the above we have

\[
G(s) = \frac{1 - \sum_{k=0}^{\infty} (1/2)^k (-s^2)^k}{s} = -\sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} s^{2k-1}
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}(2k-1)} \binom{2k-1}{k} s^{2k-1}
\]

So the distribution of $N_{10}$ is

\[
P(N_{10} = 2k - 1) = \frac{1}{2^{2k-1}(2k-1)} \binom{2k-1}{k}
\]

\[
P(N_{10} = 2k) = 0
\]

for $k = 0, 1, 2, \ldots$

\[\text{Lecture 8 - 12}\]

4.7 Branching Processes Revisit
Recall a Branching Process is a population of individuals in which
- all individuals have the same lifespan, and
- each individual will produce a random number of offsprings at the end of its life.

Let $X_n = \text{size of the } n\text{th generation}, n = 0, 1, 2, \ldots$. Let $Z_{n,i} = \#$ of offsprings produced by the $i$th individuals in the $n$th generation.

Then

\[
X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i}
\]

Suppose $Z_{n,i}$ are i.i.d with probability mass function

\[
P(Z_{n,i} = j) = P_j, \quad j \geq 0.
\]

We suppose the non-trivial case that $P_j < 1$ for all $j \geq 0$.

$\{X_n\}$ is a Markov chain with state space $\{0, 1, 2, \ldots\}$.
Generating Functions of the Branching Processes

Let \( g(s) = E[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k \) be the generating function of \( Z_{n,i} \), and \( G_n(s) \) be the generating function of \( X_n \), \( n = 0, 1, 2, \ldots \). Then \( \{ G_n(s) \} \) satisfies the following two iterative equation

\[
\begin{align*}
(i) & \quad G_{n+1}(s) = G_n(g(s)) \quad \text{for } n = 0, 1, 2, \ldots \\
(ii) & \quad G_{n+1}(s) = g(G_n(s)) \quad \text{if } X_0 = 1, \text{ for } n = 0, 1, 2, \ldots
\end{align*}
\]

**Proof of (i).**
\[
E[s^{X_{n+1}}|X_n] = \mathbb{E} \left[ \sum_{i=1}^{X_n} s^{Z_{n,i}} \right] = \prod_{i=1}^{X_n} E[s^{Z_{n,i}}] \quad (\text{by indep. of } Z_{n,i})
\]
\[
= \prod_{i=1}^{X_n} g(s) = g(s)^{X_n}
\]

From which, we have
\[
G_{n+1}(s) = E[s^{X_{n+1}}] = E[E[s^{X_{n+1}}|X_n]] = E[g(s)^{X_n}] = G_n(g(s))
\]

since \( G_n(s) = E[s^{X_n}] \).

**Proof of (ii):** HW today.

Lecture 8 - 13

Extinction Probability

Let \( \pi_0 = \lim_{n \to \infty} P(X_n = 0|X_0 = 1) = P(\text{the population will eventually die out}|X_0 = 1) \)

**Proposition**

The extinction probability \( \pi_0 \) is a root of of the equation
\[
g(s) = s
\]

where \( g(s) = \sum_{k=0}^{\infty} P_k s^k \) is the generating function of \( Z_{n,i} \).

**Proof.**
\[
\pi_0 = P(\text{population dies out}) = \sum_{j=0}^{\infty} P(\text{population dies out}|X_1 = j) P_j = \sum_{j=0}^{\infty} \pi_0 P_j = g(\pi_0)
\]

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Proposition II

If \( \mu > 1 \), there is a unique root of the equation \( g(s) = s \) in the domain \([0, 1]\), and that is the extinction probability.

**Proof.** Let \( h(s) = g(s) - s \). Observe that

\[
\begin{align*}
h(0) &= g(0) - 0 = P_0 > 0 \\
h'(0) &= g'(0) - 1 = P_1 - 1 < 0
\end{align*}
\]

Then \( \mu > 1 \Rightarrow h'(1) = \mu - 1 > 0 \)

\( \Rightarrow h(s) \) is increasing near 1

\( \Rightarrow h(1 - \delta) < h(1) = 0 \) for \( \delta > 0 \) small enough

Since \( h(s) \) is continuous in \([0, 1]\), there must be a root to \( h(s) = s \). The root is unique since

\[
h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j - 1) P_j s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1
\]

\( h(s) \) is convex in \([0, 1]\).

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Example

Suppose \( X_0 = 1 \), and \( (P_0, P_1, P_2) = (1/4, 1/2, 1/4) \). Find the distribution of \( X_2 \).

\[
g(s) = \frac{1}{4} s^0 + \frac{1}{2} s^1 + \frac{1}{4} s^2 = (1 + s)^2 / 4.
\]

Since \( X_0 = 1 \), \( G_0(s) = E[s^{X_0}] = E[s^1] = s \). From (i) we have

\[
G_1(s) = G_0(g(s)) = g(s) = (1 + s)^2 / 4
\]

\[
G_2(s) = G_1(g(s)) = \left( \frac{1}{4} \right)^2 \left( 1 + \frac{1}{4} (1 + s)^2 \right)^2 = \frac{1}{64} (5 + 2s + s^2)^2
\]

\[
= \frac{1}{64} (25 + 20s + 14s^2 + 4s^2 + s^4) = \sum_{k=0}^{\infty} P(X_2 = k)s^k
\]

The coefficient of \( s^k \) in the polynomial of \( G_2(s) \) is the chance that \( X_2 = k \).

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P(X_2 = k) )</td>
<td>25</td>
<td>20</td>
<td>14</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

and \( P(X_2 = k) = 0 \) for \( k \geq 5 \).

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Proposition I

Let \( \mu = E[Z_{n,i}] = \sum_{j=0}^{\infty} j P_j \). If \( \mu \leq 1 \), the extinction probability \( \pi_0 \) is 1 unless \( P_2 = 1 \).

**Proof.** Let \( h(s) = g(s) - s \). Since \( g(1) = 1, g'(1) = \mu, \)

\[
h(1) = g(1) - 1 = 0,
\]

\[
h'(s) = \left( \sum_{j=1}^{\infty} j P_j s^{j-1} \right) - 1 \leq \left( \sum_{j=1}^{\infty} j P_j \right) - 1 = \mu - 1 \quad \text{for } 0 \leq s < 1
\]

Thus \( \mu \leq 1 \Rightarrow h'(s) \leq 0 \) for \( 0 \leq s < 1 \)

\( \Rightarrow h(s) \) is non-increasing in \([0, 1]\)

\( \Rightarrow h(s) > h(1) = 0 \) for \( 0 \leq s < 1 \)

\( \Rightarrow g(s) > s \) for \( 0 \leq s < 1 \)

\( \Rightarrow \) There is no root in \([0, 1]\).

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