STAT253/317 Winter 2014 Lecture 8

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Generating Functions

Generating Functions

For a non-negative-integer-valued random variable T, the generating function of T is the expected value of s^T

$$G(s) = \mathrm{E}[s^T] = \sum\nolimits_{k=0}^{\infty} s^k \mathrm{P}(T = k),$$

in which s^T is defined as 0 if $T = \infty$.

- $G(s)$ is a power series converging absolutely for all $-1 < s < 1$.
- $G(1) = \mathrm{P}(T < \infty) \begin{cases} = 1 & \text{if } T \text{ is finite w/ prob. } 1 \\ < 1 & \text{otherwise} \end{cases}$
- $$P(T = k) = \frac{G^{(k)}(0)}{k!}$$

  Knowing $G(s) \Leftrightarrow$ Knowing $\mathrm{P}(T = k)$ for all $k = 0, 1, 2, \ldots$

More Properties of Generating Functions

$$G(s) = \mathrm{E}[s^T] = \sum_{k=0}^{\infty} s^k \mathrm{P}(T = k)$$

- $\mathrm{E}[T] = \lim_{s \to 1^-} G'(s)$ if it exists because

$$G'(s) = \frac{d}{ds}\mathrm{E}[s^T] = \mathrm{E}[Ts^{T-1}] = \sum_{k=1}^{\infty} s^{k-1} k \mathrm{P}(T = k).$$

- $\mathrm{E}[T(T-1)] = \lim_{s \to 1^-} G''(s)$ if it exists because

$$G''(s) = \mathrm{E}[T(T-1)s^{T-2}] = \sum\nolimits_{k=2}^{\infty} s^{k-2} k(k-1) \mathrm{P}(T = k)$$

- If T and U are independent non-negative-integer-valued random variables, with generating function $G_T(s)$ and $G_U(s)$ respectively, then the generating function of $T + U$ is

$$G_{T+U}(s) = \mathrm{E}[s^{T+U}] = \mathrm{E}[s^T]\mathrm{E}[s^U] = G_T(s)G_U(s)$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p,\ P_{i,i-1} = 1 - p = q,$ for $i = 1, 2, 3 \ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned} N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{time to reach state } j \text{ starting in state } i \end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, $N_{01}, N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i+1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i-1 \end{cases} \qquad (1)$$

  where $N^*_{i-1,i} \sim N_{i-1,i}$, $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i-1,i}$, $N^*_{i,i+1}$ are indep.

Generating Function of $N_{i,i+1}$

Let $G_i(s)$ be the generating function of $N_{i,i+1}$. From (1), and by the independence of $N^*_{i-1,i}$ and $N^*_{i,i+1}$, we get that

$$G_i(s) = ps + q\mathrm{E}[s^{1+N^*_{i-1,i}+N^*_{i,i+1}}] = ps + qsG_{i-1}(s)G_i(s)$$

So

$$G_i(s) = \frac{ps}{1 - qsG_{i-1}(s)} \qquad (2)$$

Since N_{01} is always 1, we have $G_0(s) = s$. Using the iterating relation (2), we can find

$$G_1(s) = \frac{ps}{1 - qsG_0(s)} = \frac{ps}{1 - qs^2} = ps\sum_{k=0}^{\infty}(qs^2)^k = \sum_{k=0}^{\infty} pq^k s^{2k+1}$$

So $\mathrm{P}(N_{12} = n) = \begin{cases} pq^k & \text{if } n = 2k+1 \text{ for } k = 0, 1, 2 \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$

Similarly,

$$\begin{aligned} G_2(s) &= \frac{ps}{1 - qsG_1(s)} = \frac{ps(1 - qs^2)}{1 - q(1+p)s^2} \\ &= \frac{ps}{1 - q(1+p)s^2} - \frac{pqs^3}{1 - q(1+p)s^2} \\ &= ps\sum\nolimits_{k=0}^{\infty}(q(1+p)s^2)^k - pqs^3\sum\nolimits_{k=0}^{\infty}(q(1+p)s^2)^k \\ &= \sum\nolimits_{k=0}^{\infty} pq^k(1+p)^k s^{2k+1} - \sum\nolimits_{k=0}^{\infty} pq^{k+1}(1+p)^k s^{2k+3} \\ &= ps + \sum\nolimits_{k=1}^{\infty} pq^k[(1+p)^k - (1+p)^{k-1}]s^{2k+1} \\ &= ps + \sum\nolimits_{k=1}^{\infty} p^2q^k(1+p)^{k-1}s^{2k+1} \end{aligned}$$

So

$$\mathrm{P}(N_{23} = n) = \begin{cases} p & \text{if } n = 1 \\ p^2q^k(1+p)^{k-1} & \text{if } n = 2k+1 \text{ for } k = 1, 2, \ldots \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Mean of $N_{i,i+1}$

Recall that $G_i'(1) = E(N_{i,i+1})$. Let $m_i = E(N_{i,i+1}) = G_i'(1)$.

$$G_i'(s) = \frac{p(1 - qsG_{i-1}(s)) + ps(qG_{i-1}(s) + qsG_{i-1}'(s))}{(1 - qsG_{i-1}(s))^2}$$
$$= \frac{p + pqs^2 G_{i-1}'(s)}{(1 - qsG_{i-1}(s))^2}$$

Since $N_{i,i+1} < \infty$, $G_i(1) = 1$ for all $i = 0, 1, \ldots, n-1$. We have

$$m_i = G_i'(1) = \frac{p + pqG_{i-1}'(1)}{(1-q)^2} = \frac{1 + qG_{i-1}'(1)}{p} = \frac{1}{p} + \frac{q}{p} m_{i-1}$$
$$= \frac{1}{p} + \frac{q}{p}\left(\frac{1}{p} + \frac{q}{p} m_{i-2}\right)$$
$$= \frac{1}{p}\left[1 + \frac{q}{p} + \left(\frac{q}{p}\right)^2 + \ldots + \left(\frac{q}{p}\right)^{i-1}\right] + \left(\frac{q}{p}\right)^i m_0$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + \left(\frac{q}{p}\right)^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Example: Symmetric Random Walk on $(-\infty, \infty)$

State space $= \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$$P_{i,i+1} = P_{i,i-1} = 1/2, \quad \text{for all integer } i$$

- 1 classes, recurrent, null-recurrent or positive-recurrent?
- For i, j, define $N_{ij} = \min\{m > 0 : X_m = j | X_0 = i\}$.
- Note $N_{00} = 1 + N_{10}$
- Given $X_0 = 1$

$$N_{10} = \begin{cases} 1 & \text{if } X_1 = 0 \\ 1 + N_{21} + N_{10}' & \text{if } X_1 = 2 \end{cases} \tag{3}$$

Note N_{21} and N_{10}' are independent and have the same distribution as N_{10} (Why?)

Generating Function of N_{10}

Let $G(s)$ be the generating function of N_{10}. From (3), we know that

$$G(s) = \frac{1}{2}s + \frac{1}{2}\mathrm{E}[s^{1+N_{21}+N_{10}'}] = \frac{1}{2}s + \frac{1}{2}sG(s)^2$$

which is a quadratic equation in $G(s)$. The two roots are

$$G(s) = \frac{1 \pm \sqrt{1 - s^2}}{s}$$

Since $G(s)$ must lie between 0 and 1 when $0 < s < 1$. So

$$G(s) = \frac{1 - \sqrt{1 - s^2}}{s}$$

Note that

$$G'(s) = \frac{1}{\sqrt{1 - s^2} + 1 - s^2}, \quad \mathrm{E}[N_{10}] = \lim_{s \to 1^-} G'(s) = \infty$$

which implies $\mathrm{E}[N_{00}] = 1 + \mathrm{E}[N_{10}] = \infty$.
$\Rightarrow$ Symmetric random walk is null recurrent.

The power series expansion of $G(s) = \frac{1 - \sqrt{1-s^2}}{s}$ can be found via Newton's binomial formula

$$(1 - s^2)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-s^2)^k$$

where $\binom{\alpha}{0} = 1$ and for $k \geq 1$, $\binom{\alpha}{k} = \prod_{i=0}^{k-1} (\alpha - i)/k!$.

$$\binom{1/2}{k} = \frac{\frac{1}{2}(\frac{1}{2} - 1)(\frac{1}{2} - 2)(\frac{1}{2} - 3)\ldots(\frac{1}{2} - k + 1)}{k!}$$
$$= \frac{(-1)^{k-1} 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2k-3)}{2^k k!}$$
$$= \frac{(-1)^{k-1} 1 \cdot 1 \cdot 3 \cdot 5 \ldots (2k-3)(2k-1)}{2^k k!(2k-1)}$$
$$= \frac{(-1)^{k-1} \frac{(2k-1)!}{2^{k-1}(k-1)!}}{2^k k!(2k-1)} = \frac{(-1)^{k-1}}{2^{2k-1}(2k-1)} \binom{2k-1}{k}$$

From all the above we have

$$G(s) = \frac{1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-s^2)^k}{s} = -\sum_{k=1}^{\infty} (-1)^k \binom{1/2}{k} s^{2k-1}$$
$$= \sum_{k=1}^{\infty} \frac{1}{2^{2k-1}(2k-1)} \binom{2k-1}{k} s^{2k-1}$$

So the distribution of N_{10} is

$$\mathrm{P}(N_{10} = 2k - 1) = \frac{1}{2^{2k-1}(2k-1)} \binom{2k-1}{k}$$
$$\mathrm{P}(N_{10} = 2k) = 0$$

for $k = 0, 1, 2, \ldots$

4.7 Branching Processes Revisit

Recall a Branching Process is a population of individuals in which

- all individuals have the same lifespan, and
- each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the nth generation, $n = 0, 1, 2, \ldots$. Let $Z_{n,i} = \#$ of offsprings produced by the ith individuals in the nth generation. Then

$$X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i} \tag{4}$$

Suppose $Z_{n,i}$'s are i.i.d with probability mass function

$$P(Z_{n,i} = j) = P_j, \; j \geq 0.$$

We suppose the non-trivial case that $P_j < 1$ for all $j \geq 0$.
$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Generating Functions of the Branching Processes

Let $g(s) = \mathrm{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k$ be the generating function of $Z_{n,i}$, and $G_n(s)$ be the generating function of X_n, $n = 0, 1, 2, \ldots$. Then $\{G_n(s)\}$ satisfies the following two iterative equation

$$\text{(i) } G_{n+1}(s) = G_n(g(s)) \quad \text{for } n = 0, 1, 2, \ldots$$
$$\text{(ii) } G_{n+1}(s) = g(G_n(s)) \quad \text{if } X_0 = 1, \text{ for } n = 0, 1, 2, \ldots$$

Proof of (i).

$$\begin{aligned}\mathrm{E}[s^{X_{n+1}}|X_n] &= \mathrm{E}\left[s^{\sum_{i=1}^{X_n} Z_{n,i}}\right] \mathrm{E}\left[\prod_{i=1}^{X_n} s^{Z_{n,i}}\right] \\ &= \prod_{i=1}^{X_n} \mathrm{E}[s^{Z_{n,i}}] \quad \text{(by indep. of } Z_{n,i}) \\ &= \prod_{i=1}^{X_n} g(s) = g(s)^{X_n}\end{aligned}$$

From which, we have

$$G_{n+1}(s) = \mathrm{E}[s^{X_{n+1}}] = \mathrm{E}[\mathrm{E}[s^{X_{n+1}}|X_n]] = \mathrm{E}[g(s)^{X_n}] = G_n(g(s))$$

since $G_n(s) = \mathrm{E}[s^{X_n}]$.

Proof of (ii): HW today

Example

Suppose $X_0 = 1$, and $(P_0, P_1, P_2) = (1/4, 1/2, 1/4)$. Find the distribution of X_2.

Sol.

$$g(s) = \frac{1}{4}s^0 + \frac{1}{2}s^1 + \frac{1}{4}s^2 = (1+s)^2/4.$$

Since $X_0 = 1$, $G_0(s) = \mathrm{E}[s^{X_0}] = \mathrm{E}[s^1] = s$. From (i) we have

$$\begin{aligned}G_1(s) &= G_0(g(s)) = g(s) = (1+s)^2/4 \\ G_2(s) &= G_1(g(s)) = \frac{1}{4}(1 + \frac{1}{4}(1+s)^2)^2 = \frac{1}{64}(5 + 2s + s^2)^2 \\ &= \frac{1}{64}(25 + 20s + 14s^2 + 4s^2 + s^4) = \sum_{k=0}^{\infty} \mathrm{P}(X_2 = k)s^k\end{aligned}$$

The coefficient of s^k in the polynomial of $G_2(s)$ is the chance that $X_2 = k$.

k	0	1	2	3	4
$\mathrm{P}(X_2 = k)$	$\frac{25}{64}$	$\frac{20}{64}$	$\frac{14}{64}$	$\frac{4}{64}$	$\frac{1}{64}$

and $\mathrm{P}(X_2 = k) = 0$ for $k \geq 5$.

Extinction Probability

$$\begin{aligned}\text{Let } \pi_0 &= \lim_{n\to\infty} \mathrm{P}(X_n = 0|X_0 = 1) \\ &= \mathrm{P}(\text{the population will eventually die out}|X_0 = 1)\end{aligned}$$

Proposition

The extinction probability π_0 is a root of of the equation

$$g(s) = s \qquad (5)$$

where $g(s) = \sum_{k=0}^{\infty} P_k s^k$ is the generating function of $Z_{n,i}$.

Proof.

$$\begin{aligned}\pi_0 &= \mathrm{P}(\text{population dies out}) \\ &= \sum_{j=0}^{\infty} \mathrm{P}(\text{population dies out}|X_1 = j)P_j \\ &= \sum_{j=0}^{\infty} \pi_0^j P_j = g(\pi_0)\end{aligned}$$

Proposition I

Let $\mu = \mathrm{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. If $\mu \leq 1$, the extinction probability π_0 is 1 unless $P_1 = 1$.

Proof. Let $h(s) = g(s) - s$. Since $g(1) = 1$, $g'(1) = \mu$,

$$h(1) = g(1) - 1 = 0,$$
$$h'(s) = \Big(\sum_{j=1}^{\infty} jP_j s^{j-1}\Big) - 1 \leq \Big(\sum_{j=1}^{\infty} jP_j\Big) - 1 = \mu - 1 \quad \text{for } 0 \leq s < 1$$

$$\begin{aligned}\text{Thus } \mu \leq 1 &\Rightarrow h'(s) \leq 0 \text{ for } 0 \leq s < 1 \\ &\Rightarrow h(s) \text{ is non-increasing in } [0,1) \\ &\Rightarrow h(s) > h(1) = 0 \text{ for } 0 \leq s < 1 \\ &\Rightarrow g(s) > s \qquad \text{for } 0 \leq s < 1 \\ &\Rightarrow \text{ There is no root in [0,1).}\end{aligned}$$

Proposition II

If $\mu > 1$, there is a unique root of the equation $g(s) = s$ in the domain $[0, 1)$, and that is the extinction probability.

Proof. Let $h(s) = g(s) - s$. Observe that

$$\begin{aligned}h(0) &= g(0) = P_0 > 0 \\ h'(0) &= g'(0) - 1 = P_1 - 1 < 0\end{aligned}$$

$$\begin{aligned}\text{Then } \mu > 1 &\Rightarrow h'(1) = \mu - 1 > 0 \\ &\Rightarrow h(s) \text{ is increasing near 1} \\ &\Rightarrow h(1-\delta) < h(1) = 0 \text{ for } \delta > 0 \text{ small enough}\end{aligned}$$

Since $h(s)$ is continuous in $[0, 1)$, there must be a root to $h(s) = s$. The root is unique since

$$h''(s) = g''(s) = \sum_{j=2}^{\infty} j(j-1)P_j s^{j-2} \geq 0 \quad \text{for } 0 \leq s < 1$$

$h(s)$ is convex in [0,1).