STAT253/317 Winter 2014 Lecture 7

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- • The Trick of One-Step Conditioning
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The Trick of One-Step Conditioning

Many Markov chains $\{X_n\}$ have some iterative relationships between consecutive terms, e.g.,

$$X_{n+1} = g(X_n, \xi_{n+1}) \quad \text{for all } n$$

where $\{\xi_n, n = 0, 1, 2, \ldots\}$ are some i.i.d. random variables and X_n is independent of $\{\xi_k : k > n\}$.

In many cases, we can use the iterative relationship to find $\mathbb{E}[X_n]$ and $\mathrm{Var}[X_n]$ without knowing the distribution of X_n.

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n])\end{aligned}$$

Example 1: Simple Random Walk

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with prob } p \\ X_n - 1 & \text{with prob } q = 1 - p \end{cases}$$

So

$$\begin{aligned}\mathbb{E}[X_{n+1}|X_n] &= p(X_n + 1) + q(X_n - 1) = X_n + p - q \\ \mathrm{Var}[X_{n+1}|X_n] &= 4pq\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[X_{n+1}] &= \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = \mathbb{E}[X_n] + p - q \\ \mathrm{Var}(X_{n+1}) &= \mathbb{E}[\mathrm{Var}(X_{n+1}|X_n)] + \mathrm{Var}(\mathbb{E}[X_{n+1}|X_n]) \\ &= \mathbb{E}[4pq] + \mathrm{Var}(X_n + p - q) = 4pq + \mathrm{Var}(X_n)\end{aligned}$$

So

$$\mathbb{E}[X_n] = n(p - q) + \mathbb{E}[X_0], \qquad \mathrm{Var}(X_n) = 4npq + \mathrm{Var}(X_0)$$

Example 2: Ehrenfest Urn Model with M Balls

Recall that

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } \frac{M - X_n}{M} \\ X_n - 1 & \text{with probability } \frac{X_n}{M} \end{cases}$$

We have

$$\mathbb{E}[X_{n+1}|X_n] = (X_n{+}1)\times\frac{M - X_n}{M} + (X_n{-}1)\times\frac{X_n}{M} = 1 + \left(1 - \frac{2}{M}\right) X_n.$$

Thus

$$\mathbb{E}[X_{n+1}] = \mathbb{E}[\mathbb{E}[X_{n+1}|X_n]] = 1 + \left(1 - \frac{2}{M}\right)\mathbb{E}[X_n]$$

Subtracting $M/2$ from both sided of the equation in (a), we have

$$\mathbb{E}[X_{n+1}] - \frac{M}{2} = \left(1 - \frac{2}{M}\right)(\mathbb{E}[X_n] - \frac{M}{2})$$

Thus

$$\mathbb{E}[X_n] - \frac{M}{2} = \left(1 - \frac{2}{M}\right)^n (\mathbb{E}[X_0] - \frac{M}{2})$$

Example 3: Branching Processes (Section 4.7)

Consider a population of individuals.

- All individuals have the same lifetime
- Each individual will produce a random number of offsprings at the end of its life

Let X_n = size of the n-th generation, $n = 0, 1, 2, \ldots$.
If $X_{n-1} = k$, the k individuals in the $(n-1)$-th generation will independently produce $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,k}$ new offsprings, and $Z_{n,1}, Z_{n,2}, \ldots, Z_{n,X_{n-1}}$ are i.i.d such that

$$P(Z_{n,i} = j) = P_j,\ j \geq 0.$$

We suppose that $P_j < 1$ for all $j \geq 0$.

$$X_n = \sum\nolimits_{i=1}^{X_{n-1}} Z_{n,i} \tag{1}$$

$\{X_n\}$ is a Markov chain with state space $= \{0, 1, 2, \ldots\}$.

Mean of a Branching Process

Let $\mu = \mathbb{E}[Z_{n,i}] = \sum_{j=0}^{\infty} jP_j$. Since $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = \mathbb{E}\left[\sum\nolimits_{i=1}^{X_{n-1}} Z_{n,i}\middle|X_{n-1}\right] = X_{n-1}\mathbb{E}[Z_{n,i}] = X_{n-1}\mu$$

So

$$\mathbb{E}[X_n] = \mathbb{E}[\mathbb{E}[X_n|X_{n-1}]] = \mathbb{E}[X_{n-1}\mu] = \mu\mathbb{E}[X_{n-1}]$$

If $X_0 = 1$, then

$$\mathbb{E}[X_n] = \mu\mathbb{E}[X_{n-1}] = \mu^2\mathbb{E}[X_{n-2}] = \ldots = \mu^n\mathbb{E}[X_0] = \mu^n$$

- If $\mu < 1 \Rightarrow \mathbb{E}[X_n] \to 0$ as $n \to \infty \Rightarrow \lim_{n\to\infty} \mathrm{P}(X_n \geq 1) = 0$ the branching processes will eventually die out.
- What if $\mu = 1$ or $\mu > 1$?

Variance of a Branching Process

Let $\sigma^2 = \mathrm{Var}[Z_{n,i}] = \sum_{j=0}^{\infty}(j-\mu)^2 P_j$. $\mathrm{Var}(X_n)$ may be obtained using the conditional variance formula

$$\mathrm{Var}(X_n) = \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] + \mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]).$$

Again from that $X_n = \sum_{i=1}^{X_{n-1}} Z_{n,i}$, we have

$$\mathbb{E}[X_n|X_{n-1}] = X_{n-1}\mu, \quad \mathrm{Var}(X_n|X_{n-1}) = X_{n-1}\sigma^2$$

and hence

$$\begin{aligned}\mathrm{Var}(\mathbb{E}[X_n|X_{n-1}]) &= \mathrm{Var}(X_{n-1}\mu) = \mu^2\mathrm{Var}(X_{n-1}) \\ \mathbb{E}[\mathrm{Var}(X_n|X_{n-1})] &= \sigma^2\mathbb{E}[X_{n-1}] = \sigma^2\mu^{n-1}.\end{aligned}$$

So

$$\begin{aligned}\mathrm{Var}(X_n) &= \sigma^2\mu^n + \mu^2\mathrm{Var}(X_{n-1}) \\ &= \sigma^2(\mu^{n-1} + \mu^n + \ldots + \mu^{2n-2}) + \mu^{2n}\mathrm{Var}(X_0) \\ &= \begin{cases} \sigma^2\mu^{n-1}\left(\frac{1-\mu^n}{1-\mu}\right) + \mu^{2n}\mathrm{Var}(X_0) & \text{if } \mu \neq 1 \\ n\sigma^2 + \mu^{2n}\mathrm{Var}(X_0) & \text{if } \mu = 1 \end{cases}\end{aligned}$$

4.5.3 Random Walk w/ Reflective Boundary at 0

- State Space $= \{0, 1, 2, \ldots\}$
- $P_{01} = 1, P_{i,i+1} = p, \ P_{i,i-1} = 1 - p = q$, for $i = 1, 2, 3\ldots$
- Only one class, irreducible
- For $i < j$, define

$$\begin{aligned}N_{ij} &= \min\{m > 0 : X_m = j | X_0 = i\} \\ &= \text{time to reach state } j \text{ starting in state } i\end{aligned}$$

- Observe that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$
  By the Markov property, $N_{01}, N_{12}, \ldots, N_{n-1,n}$ are indep.
- Given $X_0 = i$

$$N_{i,i+1} = \begin{cases} 1 & \text{if } X_1 = i+1 \\ 1 + N^*_{i-1,i} + N^*_{i,i+1} & \text{if } X_1 = i-1 \end{cases} \quad (2)$$

  where both $N^*_{i-1,i}$ and $N^*_{i,i+1} \sim N_{i,i+1}$, and $N^*_{i-1,i}$, $N^*_{i,i+1}$ are independent.

4.5.3 Random Walk w/ Reflective Boundary at 0 (Cont'd)

Let $m_i = \mathbb{E}(N_{i,i+1})$. Taking expected value on Equation (2), we get

$$m_i = \mathbb{E}[N_{i,i+1}] = 1 + q\mathbb{E}[N^*_{i-1,i}] + q\mathbb{E}[N^*_{i,i+1}] = 1 + q(m_{i-1} + m_i)$$

Rearrange terms we get $pm_i = 1 + qm_{i-1}$ or

$$\begin{aligned}m_i &= \frac{1}{p} + \frac{q}{p}m_{i-1} \\ &= \frac{1}{p} + \frac{q}{p}(\frac{1}{p} + \frac{q}{p}m_{i-2}) \\ &= \frac{1}{p}\left[1 + \frac{q}{p} + (\frac{q}{p})^2 + \ldots + (\frac{q}{p})^{i-1}\right] + (\frac{q}{p})^i m_0\end{aligned}$$

Since $N_{01} = 1$, which implies $m_0 = 1$.

$$m_i = \begin{cases} \frac{1-(q/p)^i}{p-q} + (\frac{q}{p})^i & \text{if } p \neq 0.5 \\ 2i + 1 & \text{if } p = 0.5 \end{cases}$$

Mean of $N_{0,n}$

Recall that $N_{0n} = N_{01} + N_{12} + \ldots + N_{n-1,n}$

$$\begin{aligned}\mathbb{E}[N_{0n}] &= m_0 + m_1 + \ldots + m_{n-1} \\ &= \begin{cases} \frac{n}{p-q} - \frac{2pq}{(p-q)^2}[1 - (\frac{q}{p})^n] & \text{if } p \neq 0.5 \\ n^2 & \text{if } p = 0.5 \end{cases}\end{aligned}$$

When

$p > 0.5$	$\mathbb{E}[N_{0n}] \approx \frac{n}{p-q} - \frac{2pq}{(p-q)^2}$	linear in n
$p = 0.5$	$\mathbb{E}[N_{0n}] = n^2$	quadratic in n
$p < 0.5$	$\mathbb{E}[N_{0n}] = O(\frac{2pq}{(p-q)^2}(\frac{q}{p})^n)$	exponential in n