STAT253/317 Winter 2014 Lecture 5

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4.4 Limiting Distribution II

Positive Recurrence and Null Recurrence

For a Markov chain, consider the return time to a recurrent state i

$$T_i = \min\{n > 0 : X_n = i | X_0 = i\}$$

We say a state i is

- **positive recurrent** if $\mathbb{E}[T_i] < \infty$.
- **null recurrent** if $\mathrm{P}(T_i < \infty) = 1$ but $\mathbb{E}[T_i] = \infty$.
- **transient** if $\mathrm{P}(T_i < \infty) < 1$

We say a state is **ergodic** if it is aperiodic and positive recurrent.

In fact, positive/null recurrence is also a class property.
Proof. Homework today ([IPM10e] Exercise 4.38, 4.39).

The Fundamental Limit Theorem of Markov Chains I

Consider a recurrent irreducible aperiodic Markov chain. Then

$$\lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_j]}$$

In other words, a recurrent state j is null recurrent iff

$$\lim_{n\to\infty} P_{ii}^{(n)} = 0.$$

Moreover, if a Markov chain is irreducible and ergodic,

$$\pi_i = \lim_{n\to\infty} P_{ij}^{(n)} = \frac{1}{\mathbb{E}[T_i]}$$

is uniquely determined by the set of equations

$$\pi_i \geq 0, \quad \sum\nolimits_{i\in\mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum\nolimits_{i\in\mathfrak{X}} \pi_i P_{ij}$$

Proof. See Theorem 1.1, 1.2, 1.3 on p.81-86 in Karlin & Taylor (1975).

Define

$$I_j^{(n)} = \begin{cases} 1 & \text{if } X_n = j \\ 0 & \text{if } X_n \neq j \end{cases}, \quad n \geq 0,\ j \in \mathfrak{X}.$$

The expected **long-run proportion of time** that the Markov chain is in state j is

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=1}^{n} I_j^{(k)} \middle| X_0 = i\right] = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left[I_j^{(k)} \middle| X_0 = i\right] = \frac{1}{n}\sum_{k=1}^{n} P_{ij}^{(k)}.$$

Fact in Analysis: If $\lim_{n\to\infty} a_n = a$, then $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_k = a$.

If the limiting distribution $\lim_{n\to\infty} P_{ij}^{(k)} = \pi_j$ exists, the fact above implies the expected long-run proportion of time that the process in state j is

$$\lim_{n\to\infty} \frac{1}{n} \sum\nolimits_{k=1}^{n} P_{ij}^{(k)} = \pi_j$$

In fact, π_j is also the long-run proportion of time that the process will be in state j.

$$\frac{1}{n}\sum\nolimits_{k=1}^{n} I_j^{(k)} \to \pi_j \quad \text{as } n \to \infty$$

Why $\pi_i = 1/\mathbb{E}(T_i)$?

Consider a Markov chain started from state j. Let S_k be the time till the k-th visit to state i. Then

$$S_k = T_{ji} + T_{ii}(1) + \ldots + T_{ii}(k-1)$$

Here

- T_{ji} = the time the chain first visit state i, and
- $T_{ii}(m)$ = the time between the mth and $(m+1)$st visit to state i.

Observe that $T_{ii}(1), T_{ii}(2), \ldots T_{ii}(k-1)$ are i.i.d. and have the same distribution as T_i.
For k large, the Law of Large Numbers tells us

$$\frac{1}{k}[T_{ji} + T_{ii}(1) + T_{ii}(2) + \cdots + T_{ii}(k-1)] \approx \mathbb{E}(T_i)$$

i.e., the chain visits state i about k times in $k\mathbb{E}(T_i)$ steps.
We have just seen that in n steps, we expect about $n\pi(i)$ visits to the state i. Hence setting $n = k\mathbb{E}(T_i)$, we get the relation

$$\pi_i = 1/\mathbb{E}(T_i).$$

Remark

From the result in the previous page, we can see that if a state i is **null recurrent**, i.e., $\mathbb{E}(T_i) = \infty$, then

$$\lim_{n\to\infty} P_{ji}^{(n)} = 0, \quad \text{for all } j \in \mathfrak{X}$$

Finite-State Markov Chains Have No Null Recurrent States

In a finite-state Markov chain all recurrent states are positive recurrent.

Proof. It suffices to consider irreducible Markov chains only since a Markov chain restricted to one of its recurrent class is also a Markov chain.

Recall an irreducible Markov chain must be recurrent. Also recall that positive/null recurrence is a class property. Thus if one state is null recurrent, then all states are null recurrent. However, since $\sum_{j \in X} P_{ij}^{(n)} = 1$. As there are only finite number of states, it is impossible that $\lim_{n\to\infty} P_{ij}^{(n)} = 0$ for all $j \in \mathfrak{X}$. Thus no state can be null recurrent.

- An alternative proof is in HW4, Problem 2
- For a finite state Markov chain, a limiting distribution exists if it is irreducible and aperiodic
- [IPM10e] Exercise 4.10, 4.24, 4.46 (See HW4 for solutions)

The Fundamental Limit Theorem of Markov Chain II (⋆⋆⋆⋆⋆)

If the Markov chain is **irreducible**, then there will be a solution to

$$\pi_i \geq 0, \quad \sum_{i \in \mathfrak{X}} \pi_i = 1, \quad \pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$$

if and only if the Markov chain is **positive recurrent**.

If a solution exists then

- it will be unique, and
- $$\pi_j = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P_{ij}^{(k)} & \text{if the chain is periodic} \\ \lim_{n\to\infty} P_{ij}^{(n)} & \text{if the chain is aperiodic} \end{cases}$$

Example 1: One-Dimensional Random Walk

In Lecture 4, we have shown that 1-dim symmetric random walk has no stationary distribution.

- Conclusion: 1-dim symmetric random walk is null recurrent, i.e.
$$\mathbb{E}[T_i] = \infty \quad \text{for all state } i$$

In fact, in Lecture 3 we have shown that

$$P_{ii}^{(n)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \binom{n}{n/2} \left(\frac{1}{2}\right)^n \approx \sqrt{\frac{2}{\pi n}} & \text{if } n \text{ is even} \end{cases}$$

Thus $\pi_i = \lim_{n\to\infty} P_{ii}^{(n)} = 0$, and hence $\mathbb{E}[T_i] = 1/\pi_i = \infty$.

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\begin{aligned} P_{i,i+1} &= p \quad \text{for all } i = 0, 1, 2, \ldots \\ P_{i,i-1} &= 1 - p \quad \text{for all } i = 1, 2, \ldots \\ p_{00} &= 1 - p \end{aligned}$$

Try to solve $\pi_j = \sum_{i \in \mathfrak{X}} \pi_i P_{ij}$

$$\begin{aligned} \pi_0 &= \pi_0 P_{00} + \pi_1 P_{10} = (1-p)(\pi_0 + \pi_1) \Rightarrow \pi_1 = \tfrac{p}{1-p}\pi_0 \\ \pi_1 &= \pi_0 P_{01} + \pi_2 P_{21} = p\pi_0 + (1-p)\pi_2 \Rightarrow \pi_2 = \left(\tfrac{p}{1-p}\right)^2 \pi_0 \\ \pi_2 &= \pi_0 P_{12} + \pi_3 P_{32} = p\pi_1 + (1-p)\pi_3 \Rightarrow \pi_3 = \left(\tfrac{p}{1-p}\right)^3 \pi_0 \\ &\vdots \\ \pi_j &= p\pi_{j-1} + (1-p)\pi_{j+1} \Rightarrow \pi_{j+1} = \left(\tfrac{p}{1-p}\right)^{j+1} \pi_0 \end{aligned}$$

Ex 2: 1-D Random Walk w/ Partially Reflective Boundary

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} \left(\frac{p}{1-p}\right)^i = \begin{cases} \pi_0 \left(\frac{1-p}{1-2p}\right) & \text{if } p < 1/2 \\ \infty & \text{if } p \geq 1/2 \end{cases}$$

Conclusion: The process is positive recurrent iff $p < 1/2$, in which case

$$\pi_i = \frac{1-2p}{1-p} \left(\frac{p}{1-p}\right)^i, \quad i = 0, 1, 2, \ldots$$

Proposition 4.3 (LLN for Markov Chain)

Let $\{X_n, n \geq 1\}$ be an irreducible Markov chain with stationary probabilities π_j, $j \geq 0$, and let r be a bounded function on the state space. Then, with probability 1,

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} r(X_n)}{N} = \sum_{j \in \mathfrak{X}} r(j)\pi_j$$

Proof. If we let $a_j(N)$ be the amount of time the Markov chain spends in state j during time periods $1, \cdots, N$, then

$$\sum_{n=1}^{N} r(X_n) = \sum_{j \in \mathfrak{X}} a_j(N) r(j)$$

Since $a_j(N)/N \to \pi_j$ the result follows from the preceding upon dividing by N and then letting $N \to \infty$.

This result is the idea behind the Markov Chain Monte Carlo (MCMC) method.