STAT253/317 Winter 2014 Lecture 2

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January 10, 2014

4.2 Chapman-Kolmogorov Equation

Joint Distribution of Random Variables in a Markov Chain

Suppose $\{X_n : n = 0, 1, 2, \ldots\}$ is a stationary Markov chain with

- state space $\mathfrak{X}$ and
- transition probabilities $\{P_{ij} : i, j \in \mathfrak{X}\}$.

Define $\pi_0(i) = \mathrm{P}(X_0 = i)$, $i \in \mathfrak{X}$ to be the marginal distribution of X_0.

What is the joint distribution of X_0, X_1, X_2?

$$\begin{aligned}
&\mathrm{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2)\\
&= \mathrm{P}(X_0 = i_0)\mathrm{P}(X_1 = i_1|X_0 = i_0)\mathrm{P}(X_2 = i_2|X_1 = i_1, X_0 = i_0)\\
&= \mathrm{P}(X_0 = i_0)\mathrm{P}(X_1 = i_1|X_0 = i_0)\mathrm{P}(X_2 = i_2|X_1 = i_1) \quad (\because \text{ Markov})\\
&= \pi_0(i_0)P_{i_0 i_1}P_{i_1 i_2}
\end{aligned}$$

In general,

$$\begin{aligned}
&\mathrm{P}(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}, X_n = i_n)\\
&= \pi_0(i_0)P_{i_0 i_1}P_{i_1 i_2}\ldots P_{i_{n-1} i_n}
\end{aligned}$$

Chapman-Kolmogorov Equation

Suppose $\{X_n\}$ is a stationary Markov chain with state space $\mathfrak{X}$.
Define the n-step transition probabilities

$$P_{ij}^{(n)} = \mathrm{P}(X_{n+k} = j|X_k = i) \quad \text{for } i, j \in \mathfrak{X} \text{ and } n, k = 0, 1, 2, \ldots$$

Then for all m, $n \geq 1$,

$$P_{ij}^{(m+n)} = \sum\nolimits_{k\in\mathfrak{X}} P_{ik}^{(m)} P_{kj}^{(n)}$$

Proof.

$$\begin{aligned}
P_{ij}^{(m+n)} &= \mathrm{P}(X_{m+n} = j|X_0 = i)\\
&= \sum\nolimits_{k\in\mathfrak{X}} \mathrm{P}(X_{m+n} = j, X_n = k|X_0 = i)\\
&= \sum\nolimits_{k\in\mathfrak{X}} \mathrm{P}(X_{m+n} = j|X_n = k, X_0 = i)\mathrm{P}(X_n = k|X_0 = i)\\
&= \sum\nolimits_{k\in\mathfrak{X}} \mathrm{P}(X_{m+n} = j|X_n = k)\mathrm{P}(X_n = k|X_0 = i) \quad (\because \text{ Markov})\\
&= \sum\nolimits_{k\in\mathfrak{X}} P_{ik}^{(m)} P_{kj}^{(n)}
\end{aligned}$$

Chapman-Kolmogorov Equation in Matrix Notation

For $n = 1, 2, 3, \ldots$, let

$$\mathbb{P}^{(n)} = \begin{pmatrix}
P_{00}^{(n)} & P_{01}^{(n)} & P_{02}^{(n)} & \cdots & P_{0j}^{(n)} & \cdots\\
P_{10}^{(n)} & P_{11}^{(n)} & P_{12}^{(n)} & \cdots & P_{1j}^{(n)} & \cdots\\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots\\
P_{i0}^{(n)} & P_{i1}^{(n)} & P_{i2}^{(n)} & \cdots & P_{ij}^{(n)} & \cdots\\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{pmatrix}$$

be the n-step transition probability matrix.
The Chapman-Kolmogorov equation just asserts that

$$\mathbb{P}^{(m+n)} = \mathbb{P}^{(m)} \times \mathbb{P}^{(n)}$$

Note $\mathbb{P}^{(1)} = \mathbb{P}$, $\Rightarrow \mathbb{P}^{(2)} = \mathbb{P}^{(1)} \times \mathbb{P}^{(1)} = \mathbb{P} \times \mathbb{P} = \mathbb{P}^2$.
By induction,

$$\mathbb{P}^{(n)} = \mathbb{P}^{(n-1)} \times \mathbb{P}^{(1)} = \mathbb{P}^{n-1} \times \mathbb{P} = \mathbb{P}^n$$

Define $\pi_n(i) = \mathrm{P}(X_n = i)$, $i \in \mathfrak{X}$ to be the marginal distribution of X_n, $n = 1, 2, \ldots$. Then again by the law of total probabilities,

$$\begin{aligned}
\pi_n(j) &= \mathrm{P}(X_n = j)\\
&= \sum\nolimits_{k\in\mathfrak{X}} \mathrm{P}(X_0 = k)\mathrm{P}(X_n = j|X_0 = k) \qquad (1)\\
&= \sum\nolimits_{k\in\mathfrak{X}} \pi_0(k)P_{kj}^{(n)}
\end{aligned}$$

Suppose the state space $\mathfrak{X}$ is $\{0, 1, 2, \ldots\}$.
If we write the marginal distribution of X_n as a row vector

$$\pi_n = (\pi_n(0), \pi_n(1), \pi_n(2), \ldots),$$

then the equation (1) is

$$\pi_n = \pi_0\mathbb{P}^{(n)} = \pi_0\mathbb{P}^n$$

Example: Ehrenfest Model, 4 Balls

$$\mathbb{P} = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}
\begin{array}{c}
\begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\
\begin{pmatrix}
0 & 4/4 & 0 & 0 & 0\\
1/4 & 0 & 3/4 & 0 & 0\\
0 & 2/4 & 0 & 2/4 & 0\\
0 & 0 & 3/4 & 0 & 1/4\\
0 & 0 & 0 & 4/4 & 0
\end{pmatrix}
\end{array}$$

Q1 Find $\mathrm{P}(X_4 = 2|X_0 = 4)$

Q2 Given $\mathrm{P}(X_0 = i) = 1/5$ for $i = 0, 1, 2, 3, 4$, find $\mathrm{P}(X_4 = 2)$

Q3 Find $\mathrm{P}(X_4 = 2, X_3 \geq 2, X_2 \geq 2, X_1 \geq 2|X_0 = 4)$

$$\mathbb{P}^2 = \mathbb{P}\times\mathbb{P} = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 1/4 & 0 & 3/4 & 0 & 0\\ 0 & 5/8 & 0 & 3/8 & 0\\ 1/8 & 0 & 3/4 & 0 & 1/8\\ 0 & 3/8 & 0 & 5/8 & 0\\ 0 & 0 & 3/4 & 0 & 1/4 \end{pmatrix}\end{array}$$

$$\mathbb{P}^3 = \mathbb{P}\times\mathbb{P}^2 = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 0 & 5/8 & 0 & 3/8 & 0\\ 5/32 & 0 & 3/4 & 0 & 3/32\\ 0 & 1/2 & 0 & 1/2 & 0\\ 3/32 & 0 & 3/4 & 0 & 5/32\\ 0 & 3/8 & 0 & 5/8 & 0 \end{pmatrix}\end{array}$$

$$\mathbb{P}^4 = \mathbb{P}^2\times\mathbb{P}^2 = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32\\ 0 & 17/32 & 0 & 15/32 & 0\\ 1/8 & 0 & 3/4 & 0 & 1/8\\ 0 & 15/32 & 0 & 5/32 & 0\\ 3/32 & 0 & 3/4 & 0 & 5/32 \end{pmatrix}\end{array}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

$$\mathbb{P}^4 = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32\\ 0 & 17/32 & 0 & 15/32 & 0\\ 1/8 & 0 & 3/4 & 0 & 1/8\\ 0 & 15/32 & 0 & 5/32 & 0\\ 3/32 & 0 & 3/4\ \boxed{3/4} & 0 & 5/32 \end{pmatrix}\end{array}$$

$$\| \quad P_{42}^{(4)}$$

So for Q1, $\mathrm{P}(X_4 = 2|X_0 = 4) = ?P_{42}^{(4)} = 3/4$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q2: Given $\mathrm{P}(X_0 = i) = 1/5$ for $i = 0, 1, 2, 3, 4$, find $\mathrm{P}(X_4 = 2)$.

$$\pi_0 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}).$$

$$\pi_4 = \pi_0\mathbb{P}^4 = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \begin{pmatrix} 5/32 & 0 & 3/4 & 0 & 3/32\\ 0 & 17/32 & 0 & 15/32 & 0\\ 1/8 & 0 & 3/4 & 0 & 1/8\\ 0 & 15/32 & 0 & 17/32 & 0\\ 3/32 & 0 & 3/4 & 0 & 5/32 \end{pmatrix}$$

$$\pi_4(2) = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \begin{pmatrix} 3/4\\ 0\\ 3/4\\ 0\\ 3/4 \end{pmatrix}$$

$$= \frac{1}{5}\cdot\frac{3}{4} + \frac{1}{5}\cdot 0 + \frac{1}{5}\cdot\frac{3}{4} + \frac{1}{5}\cdot 0 + \frac{1}{5}\cdot\frac{3}{4} = \frac{9}{20}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

Q3: Find $\mathrm{P}(X_4 = 2, X_3 \geq 2, X_2 \geq 2, X_1 \geq 2|X_0 = 4)$.

Tip: Create another process $\{W_n, n = 0, 1, 2, \dots\}$ with an absorbing state A

$$W_n = \begin{cases} X_n & \text{if } X_k \geq 2 \text{ for all } k = 0, 1, 2, \dots, n\\ A & \text{otherwise} \end{cases}$$

What is the state space of $\{W_n\}$? $\{A, 2, 3, 4\}$

Is $\{W_n\}$ a Markov chain?

$$W_{n+1} = \begin{cases} A & \text{if } W_n = A\\ W_n + 1 & \text{with prob. } \frac{4-W_n}{4} \text{ if } W_n \neq A\\ W_n - 1 & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 3 \text{ or } 4\\ A & \text{with prob. } \frac{W_n}{4} \text{ if } W_n = 2 \end{cases}$$

Yes, $\{W_n\}$ is a Markov chain.

Example: Ehrenfest Model, 4 Balls (Cont'd)

What is the transition probability of $\{W_n\}$?

$$\mathbb{P}_W = \begin{array}{c} \\ A\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 1 & 0 & 0 & 0\\ 2/4 & 0 & 2/4 & 0\\ 0 & 3/4 & 0 & 1/4\\ 0 & 0 & 4/4 & 0 \end{pmatrix}\end{array}$$

Compare with the transition probability of the original process $\{X_n\}$

$$\mathbb{P} = \begin{array}{c} \\ 0\\1\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 0 & 4/4 & 0 & 0 & 0\\ 1/4 & 0 & 3/4 & 0 & 0\\ 0 & 2/4 & 0 & 2/4 & 0\\ 0 & 0 & 3/4 & 0 & 1/4\\ 0 & 0 & 0 & 4/4 & 0 \end{pmatrix}\end{array}$$

Example: Ehrenfest Model, 4 Balls (Cont'd)

How does $\{W_n\}$ helps us to solve Q3?

Observe that

$$\mathrm{P}(X_4 = 2, X_3 \geq 2, X_2 \geq 2, X_1 \geq 2|X_0 = 4) = \mathrm{P}(W_4 = 2|W_0 = 4)$$

$$\mathbb{P}_W^4 = \begin{array}{c} \\ A\\2\\3\\4 \end{array}\!\!\begin{array}{c} \begin{array}{cccc} A & 2 & 3 & 4 \end{array}\\ \begin{pmatrix} 1 & 0 & 0 & 0\\ 44/64 & 15/64 & 0 & 5/64\\ 39/64 & 0 & 25/64 & 0\\ 12/32 & 15/32\ \boxed{15/32} & 0 & 5/32 \end{pmatrix}\end{array}$$

The answer to Q3 is $\mathrm{P}(W_4 = 4|W_0 = 4) = ?P_{W42}^{(4)} = 15/32$