STAT253/317 Winter 2014 Lecture 26

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- Quadratic Variation
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Total Variation (First-Order Variation)

For a function $f(t)$, we wish to compute the amount of up and down oscillation undergone by this function between 0 and T. Let $\Pi = \{t_0, t_1, \dots, t_n\}$ be a *partition* of $[0, T]$, which is a set of times

$$0 = t_0 < t_1 < t_2 < \dots t_n = T.$$

The mesh size of the partition is defined as

$$\|\Pi\| = \max_{0 \le j \le n-1} |t_{j+1} - t_j|.$$

The **total variation** of a function $f(t)$ on the interval $[0, T]$ is defined as

$$TV_T(f) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|.$$

Total Variation (First-Order Variation) (Cont'd)

Remark 1. If the function $f(x)$ is monotone on $[0, T]$, then the total of f on the on the interval $[0, T]$ is simply $|f(0) - f(T)|$.

Remark 2. If the function $f(x)$ is monotone increasing on $[0, c]$ and monotone decreasing on $[c, T]$, then the total of f on the on the interval $[0, T]$ is $|f(0) - f(c)| + |f(c) - f(T)|$.

The total variation of Brownian motion in $[0, T]$ is ∞ for all $T > 0$.

The proof will be given later

Quadratic Variation (Second-Order Variation)

For a function $f(t)$ defined on the interval $[0, T]$, the **quadratic variation** of $f(t)$ in $[0, T]$ is defined as

$$[f, f](T) = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2.$$

For a smooth function f on $[0, T]$ with continuous derivative f', by mean-value theorem, there exists some t_j^* between t_j and t_{j+1} such that

$$|f(t_{j+1}) - f(t_j)| = |f'(t_j^*)|(t_{j+1} - t_j).$$

The quadratic variation is then

$$\begin{aligned}
&\sum_{j=0}^{n-1} [f'(t_j)^*]^2 (t_{j+1} - t_j)^2 \\
&\le \|\Pi\| \sum_{j=0}^{n-1} [f'(t_j)^*]^2 (t_{j+1} - t_j) \to \|\Pi\| \int_0^T |f'(t)|^2 dt.
\end{aligned}$$

If $f'(t)$ is continuous, then $\int_0^T |f'(t)|^2 dt < \infty$. As the mesh size $\|\Pi\| \to 0$, the quadratic variation of f must be 0.

A Useful Result

Several proofs in this lecture use the following results.

Proposition. If $X_1, X_2, \dots, X_n \dots$ is a sequence of random variable with

$$\lim_{n \to \infty} \mathbb{E}[X_n] = c \quad \text{and} \quad \lim_{n \to \infty} \mathrm{Var}(X_n) = 0$$

then $X_n \to c$ in probability.

Quadratic Variation of Standard Brownian Motion

The quadratic variation of standard Brownian motion on the interval $[0, T]$ is T. Here T is a fixed constant.

Proof. For any partition $\Pi = \{t_0, t_1, \dots, t_n\}$, since $B(t_{j+1}) - B(t_j) \sim N(0, t_{j+1} - t_j)$ for $j = 0, 1 \dots, n-1$, and is independent of each other, we have

$$\begin{aligned}
\mathbb{E}\Big[\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2\Big] &= \sum_{j=0}^{n-1} (t_{j+1} - t_j) = T \\
\mathrm{Var}\Big(\sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2\Big) &= \sum_{j=0}^{n-1} 3(t_{j+1} - t_j)^2 \\
&\le 3T\|\Pi\| \longrightarrow 0 \quad \text{as } \|\Pi\| \to 0
\end{aligned}$$

Thus

$$\lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [B(t_{j+1}) - B(t_j)]^2 = T.$$

Proof of that Brownian Motion Has Infinite Total Variation

Suppose to the contrary that Brownian motion has finite total variation,

$$\sum_{j=0}^{n-1}[B(t_{j+1})-B(t_j)]^2 \le \max_{0\le j\le n-1}|B(t_{j+1})-B(t_j)| \underbrace{\sum_{j=0}^{n-1}|B(t_{j+1})-B(t_j)|}_{\to \text{ total variation}}$$

Since the Brownian motion path is continuous with probability 1 on $[0,T]$, it is necessarily uniformly continuous on $[0,T]$. Therefore as the mesh size $\|\Pi\|\to 0$,

$$\max_{0\le j\le n-1}|B(t_{j+1})-B(t_j)|\to 0 \text{ with prob. } 1.$$

from which we conclude that $\sum_{j=0}^{n-1}[B(t_{j+1})-B(t_j)]^2\to 0$ with probability 1. This is a contradiction to the result on the previous slide.

Review of Riemann–Stieltjes Integral

The Riemann–Stieltjes integral of a real-valued function f of a real variable with respect to a real function g is defined as the limit of the approximating sum

$$\int_a^b f(t)dg(t)=\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} f(t_j^*)[g(t_{j+1})-g(t_j)]$$

where t_j^* is in the jth subinterval $[t_{j+1},t_j]$ and $\|\Pi\|$ is the mesh size $\max_{0\le j\le n-1}|t_{j+1}-t_j|$ of the partition

$$\Pi=\{a=t_0<t_1<\ldots<t_n=b\}.$$

$\int_0^T B(t)dB(t)=?$

In Riemann–Stieltjes integral, the limit does not depend on the selection of t_j^* in the subinterval $[t_{j+1},t_j]$.

However, if $g(t)$ is not sufficiently smooth, the limit may depend on the selection of t_j^*. For example, if $f(t)=g(t)=$ the standard Brownian Motion $B(t)$, we will show that

$$\lim_{\|\Pi\|\to 0}\sum_{j=0}^{n-1} B(t_j^*)[B(t_{j+1})-B(t_j)] = \begin{cases} \frac{1}{2}B(T)^2-\frac{1}{2}T & \text{if } t_j^*=t_j & \text{(Ito integral)} \\ \frac{1}{2}B(T)^2 & \text{if } t_j^*=\frac{t_{j+1}+t_j}{2} & \text{(Stratonovich integral) (1)} \\ \frac{1}{2}B(T)^2+\frac{1}{2}T & \text{if } t_j^*=t_{j+1}, & \end{cases}$$

Which definition should we choose?

Proof of Equation (1)

Observe that

$$\sum_{j=0}^{n-1} B(t_j^*)[B(t_{j+1})-B(t_j)] = I+II$$

where

$$\begin{aligned} I &= \sum_{j=0}^{n-1}\frac{1}{2}[B(t_{j+1})+B(t_j)][B(t_{j+1})-B(t_j)] \\ &= \sum_{j=0}^{n-1}\frac{1}{2}[B(t_{j+1})^2-B(t_j)^2] \\ &= \frac{1}{2}[B(t_n)^2-B(t_0)^2]=\frac{1}{2}B(T)^2 \\ II &= \sum_{j=0}^{n-1}\{B(t_j^*)-\frac{1}{2}[B(t_{j+1})+B(t_j)]\}[B(t_{j+1})-B(t_j)] \\ &= \frac{1}{2}\sum_{j=0}^{n-1}\{B(t_j^*)-B(t_j)-[B(t_{j+1})-B(t_j^*)]\} \\ &\qquad \times [B(t_{j+1})-B(t_j^*)+B(t_j^*)-B(t_j)] \\ &= \frac{1}{2}\sum_{j=0}^{n-1}\{[B(t_j^*)-B(t_j)]^2-[B(t_{j+1})-B(t_j^*)]^2\} \end{aligned}$$

Proof of Equation (1) (Cont'd)

For $t_j^*=t_j$, observe that

$$\begin{aligned} II &= \frac{1}{2}\Big\{\sum_{j=0}^{n-1}[B(t_j^*)-B(t_j)]^2-\sum_{j=0}^{n-1}[B(t_{j+1})-B(t_j^*)]^2\Big\} \\ &= \frac{1}{2}\Big\{\sum_{j=0}^{n-1}\underbrace{[B(t_j)-B(t_j)]^2}_{=0}-\underbrace{\sum_{j=0}^{n-1}[B(t_{j+1})-B(t_j^*)]^2}_{\text{quadratic variation}}\Big\} \\ &\to \frac{1}{2}(0-T)=-\frac{T}{2} \quad \text{in probability as } \max_{0\le j\le n-1}|t_{j+1}-t_j|\to 0 \end{aligned}$$

Similarly, for $t_j^*=t_{j+1}$, observe that

$$\begin{aligned} II &= -\frac{1}{2}\Big\{\underbrace{\sum_{j=0}^{n-1}[B(t_{j+1})-B(t_j)]^2}_{\text{quadratic variation}}+\sum_{j=0}^{n-1}\underbrace{[B(t_{j+1})-B(t_{j+1})]^2}_{=0}\Big\} \\ &\to \frac{1}{2}(T-0)=\frac{T}{2} \quad \text{in probability as } \max_{0\le j\le n-1}|t_{j+1}-t_j|\to 0 \end{aligned}$$

Proof of Equation (1) (Cont'd)

For $t_j^*=(t_{j+1}+t_j)/2$,

$$II=\frac{1}{2}\Big\{\sum_{j=0}^{n-1}[B(\frac{t_{j+1}+t_j}{2})-B(t_j)]^2-\sum_{j=0}^{n-1}[B(t_{j+1})-B(\frac{t_{j+1}+t_j}{2})]^2\Big\}$$

So

$$\begin{aligned} \mathbb{E}[II] &= \frac{1}{2}\sum_{j=0}^{n-1}\frac{(t_{j+1}-t_j)}{2}-\frac{(t_{j+1}-t_j)}{2}=0 \\ \mathrm{Var}(II) &= \frac{1}{4}\sum_{j=0}^{n-1}3(\frac{t_{j+1}-t_j}{2})^2+3(\frac{t_{j+1}-t_j}{2})^2 \\ &= \frac{3}{8}\sum_{j=0}^{n-1}(t_{j+1}-t_j)^2\le\frac{3}{8}\|\Pi\|T\to 0 \text{ as } \|\Pi\|\to 0 \end{aligned}$$

The above shows $II\longrightarrow 0$ in probability as $\|\Pi\|\to 0$.