Maxwell Distribution of a Brownian Motion with drift $< 0$

Let $\{B(t), t \geq 0\}$ be a Brownian Motion with drift coefficient $\mu < 0$ and variance parameter $\sigma^2$. Consider the maximum of the process

$$W = \max_{0 \leq t < \infty} B(t)$$

Then $W$ has the exponential distribution with rate $2|\mu|/\sigma^2$,

$$P(W \geq w) = e^{-\frac{2|\mu|}{\sigma^2}w}, \quad w \geq 0.$$ 

**Proof.** Using the formula for $P(B(T_{-a}) = b)$ we derived in the previous lecture, since $\mu < 0$, $\exp(2|\mu|/\sigma^2) \to 0$, we have

$$\lim_{a \to \infty} P(B(T_{-a}) = b) = \lim_{a \to \infty} e^{\frac{2\mu a}{\sigma^2} - 1} e^{-2\mu b/\sigma^2} = e^{2\mu b/\sigma^2}$$

The left-hand side becomes the probabilities that the process ever reaches $b$, that is, the probabilities that the maximum of the process ever exceeds $b$. Thus for $w = b$, we have

$$P(W \geq w) = \exp(2|\mu|/\sigma^2) = \exp(-2|\mu|w/\sigma^2).$$

Let $T_a = \min\{ t : B(t) = a \}$.

- Though the First Wald’s identity doesn’t help, the Third identity will. For $\theta > 0$, we have

$$\mathbb{E}[e^{\theta B(T_{a}) - (\theta + \frac{\sigma^2}{2}) (T_{a})}] = 1.$$ 

- Since $\mu > 0$, $P(T_a < \infty) = 1$, $(T_a \wedge n) \to T$ as $n \to \infty$.

- By the continuity of Brownian motion path, $B(T_a \wedge n) \to B(T_a)$ as $n \to \infty$.

- Since $B(T_a \wedge n) \to B(T_a)$ for all $n$, and that $\theta > 0$, $\mu > 0$, we have

$$e^{\theta B(T_{a}) - (\theta + \frac{\sigma^2}{2}) (T_{a}) n} \leq e^{\theta B(T_{a})} \leq e^{\theta a}.$$ 

- By the Dominated Convergence Theorem,

$$1 = \lim_{n \to \infty} \mathbb{E}[e^{\theta B(T_{a}) - (\theta + \frac{\sigma^2}{2}) (T_{a}) n}] = e^{\theta a} \mathbb{E}[e^{\theta B(T_{a})}].$$

That is,

$$\mathbb{E}[e^{\theta B(T_{a})}] = e^{\theta a} \quad \text{for all } \theta > 0.$$

Let $T_a = \min\{ t : B(t) = a \}$.

- Maximum of a Brownian Motion with drift $< 0$.

- More Applications of Wald’s Identities.

Hitting Times of a Brownian Motion with Drift

Consider a Brownian motion process $\{B(t), t \geq 0\}$ with drift coefficient $\mu > 0$ and variance parameter $\sigma^2$ and consider the hitting time to the value $a > 0$

$$T_a = \min\{ t : B(t) = a \}.$$

- Reflection principle doesn’t apply to Brownian motion with drift.

- Alternative tool: Wald’s identities

- $T_a$ is a stopping time: $\{T_a \leq t\} = \{\max_{0 \leq s \leq t} B(s) \geq a\}$ depends on $\{B(s), 0 \leq s \leq t\}$ only.

- $T_a$ is finite, but unbounded, can’t apply Wald’s identities directly. Try the truncated stopping time $T_a \wedge n = \min(T_a, n)$

- First Wald’s identity $\Rightarrow \mathbb{E}[B(T_a \wedge n)] = \mu \mathbb{E}[T_a \wedge n]$

- However, $|B(T \wedge n)|$ is not uniformly bounded. We cannot use the Dominated convergence theorem to show that

$$a = \mathbb{E}[B(T_a)] = \lim_{n \to \infty} \mathbb{E}[B(T_a \wedge n)] = \lim_{n \to \infty} \mathbb{E}[T_a \wedge n] = \mu \mathbb{E}[T_a].$$

Let $T_a = \min\{ t : B(t) = a \}$.

- Make a change of variable by letting $\lambda = -((\theta \mu + \frac{\sigma^2}{2})$. Then by solving the quadratic equation

$$\sigma^2 \theta^2 + 2\mu \theta + 2\lambda = 0$$

for $\theta$, we will get

$$\theta = \frac{-\mu \pm \sqrt{\mu^2 - 2\lambda \sigma^2}}{\sigma^2}.$$ 

Since $\theta > 0$, we know that

$$\theta = \frac{-\mu + \sqrt{\mu^2 - 2\lambda \sigma^2}}{\sigma^2} > 0 \quad \text{for } \lambda \leq \frac{\mu^2}{2\sigma^2}$$

Hence, we can get the MGF for $T_a$,

$$M(\lambda) = \mathbb{E}[e^{\lambda T_a}] = \exp\left(\frac{\mu - \sqrt{\mu^2 - 2\lambda \sigma^2}}{\sigma^2} a\right), \quad \lambda \leq \frac{\mu^2}{2\sigma^2}.$$
It is possible to find the distribution with the MGF given above. The probability density of $T_2$ is

$$f(t) = \frac{a}{\sigma \sqrt{2\pi t^3}} \exp \left( -\frac{(a - \mu t)^2}{2\sigma^2 t} \right), \quad t > 0$$

**Corollary:** For a Brownian motion process $\{B(t), t \geq 0\}$ with drift coefficient $\mu > 0$ and variance parameter $\sigma^2$, the distribution of $\max_{0 \leq s \leq t} B(s)$ is

$$P \left( \max_{0 \leq s \leq t} B(s) \geq a \right) = P(T_s \leq t)$$

$$= \int_0^t \frac{a}{\sigma \sqrt{2\pi u^3}} \exp \left( -\frac{(a - \mu u)^2}{2\sigma^2 u} \right) du$$

**Example: $T_{-a,b}$ for SBM (Cont’d)**

- Since $P(T < \infty) = 1$, $(T \cap n) \rightarrow T$ as $n \rightarrow \infty$ w/ prob. 1.
- Since $-a \leq B(T \cap n) \leq b$ $\Rightarrow \{B(T \cap n)\} \leq a + b$, for all $n$
  
  $$e^{\theta B(T \cap n) - \theta^2(T \cap n)/2} \leq e^{(a+b)}.$$

- By the Dominated Convergence Theorem,

  $$1 = \lim_{n \rightarrow \infty} E\left[e^{\theta B(T \cap n) - \theta^2(T \cap n)/2}\right] = E\left[e^{\theta B(T) - \theta^2T/2}\right]$$

  $$= e^{-\theta a}E\left[e^{-\theta T/2}B(T) = -a\right]P(B(T) = -a)$$

  $$+ e^{\theta B}E\left[e^{-\theta T/2}B(T) = b\right]P(B(T) = b)$$

  $$= \frac{be^{\theta a}}{a+b} E\left[e^{-\theta T/2}B(T) = -a\right] + \frac{ae^{\theta b}}{a+b} E\left[e^{-\theta T/2}B(T) = b\right]$$

**Example: $T_{-a,b}$ for SBM (Cont’d)**

For constants $a, b > 0$, recall in Lecture 24 we let $T = T_{-a,b}$ be the first time $t$ the standard Brownian Motion process hits $-a$ or $b$

$$T_{-a,b} = \min\{t : B(t) = -a, \text{ or } B(t) = b\}$$

and used the first and second Wald’s identity to show that

$$P(B(T) = -a) = \frac{b}{a+b}, \quad P(B(T) = b) = \frac{a}{a+b}$$

and that

$$E[T] = ab.$$

Today we will use the third Wald’s identity to find the distribution of $T$.

$$E[e^{\theta(T \cap n) - \theta^2(T \cap n)/2}] = 1$$

**Example: $T_{-a,b}$ for SBM (Cont’d)**

$$a + b = be^{-\theta a}E[e^{-\theta T/2}B(T) = -a] + ae^{\theta b}E[e^{-\theta T/2}B(T) = b].$$

This equation is valid if we change $\theta$ to $-\theta$:

$$a + b = be^{\theta a}E[e^{-\theta T/2}B(T) = -a] + ae^{-\theta b}E[e^{-\theta T/2}B(T) = b].$$

Now we have 2 equations with 2 unknowns

$$x = E[e^{-\theta T/2}B(T) = -a] \quad \text{and} \quad y = E[e^{-\theta T/2}B(T) = b].$$

Solving the equations we can get that

$$x = \frac{a+b}{b} \frac{e^{\theta b} - e^{-\theta b}}{e^{(a+b)\theta} - e^{-(a+b)\theta}}, \quad y = \frac{a+b}{a} \frac{e^{\theta b} - e^{-\theta b}}{e^{(a+b)\theta} - e^{-(a+b)\theta}}$$

**Example: $T_{-a,b}$ for SBM (Cont’d)**

Thus

$$E[e^{-\theta T/2}] = E[e^{-\theta T/2}B(T) = -a]P(B(T) = -a)$$

$$+ E[e^{-\theta T/2}B(T) = -a]P(B(T) = b)$$

$$= \frac{b}{a+b}x + \frac{a}{a+b}y = \frac{e^{\theta b} - e^{-\theta b}}{e^{(a+b)\theta} - e^{-(a+b)\theta}}$$

$$+ \frac{e^{-\theta b} - e^{\theta b}}{e^{-(a+b)\theta} - e^{(a+b)\theta}}$$

$$= \frac{e^{-\theta b} + e^{\theta b}}{1 + e^{-(a+b)\theta}}$$

Let $\lambda = \theta^2/2$, we get that $\theta = \sqrt{2\lambda}$ and the moment generating function of $T$ that

$$E[e^{-\lambda T}] = \frac{e^{-a\sqrt{2\lambda}} + e^{-b\sqrt{2\lambda}}}{1 + e^{-(a+b)\sqrt{2\lambda}}}$$