Lecture 23 Hitting Time, Maximum, Reflection Principle

Hitting Times (First Passage Times)

Let $T_a = \min\{t : B(t) = a\}$ be the first time the standard Brownian motion process hits a. For $a > 0$, consider

$$\mathrm{P}(B(t) \geq a) = \mathrm{P}(B(t) \geq a | T_a \leq t)\mathrm{P}(T_a \leq t) + \underbrace{\mathrm{P}(B(t) \geq a | T_a > t)}_{=0}\mathrm{P}(T_a > t)$$

$\mathrm{P}(B(t) \geq a | T_a > t)$ is clearly 0, since the path of a Brownian motion is continuous $B(t)$ cannot surpass a without having hit a. For the 1st term, note if $T_a \leq t$, the process must have hit a by time t, and by symmetry, it is just as likely to be above a or below a at time t, i.e.,

$$\mathrm{P}(B(t) \geq a | T_a \leq t) = 1/2.$$

Thus $\mathrm{P}(T_a \leq t) = 2\mathrm{P}(B(t) \geq a) = 2 - 2\Phi(a/\sqrt{t})$, where $\Phi(x)$ is the CDF of $N(0, 1)$.

HW: Show that $\mathrm{P}(T_a < \infty) = 1$ and $\mathbb{E}[T_a] = \infty$ for $a > 0$.

Maximum

Another random variable of interest is the maximum

$$\max_{0 \leq s \leq t} B(s).$$

By continuity, we know

$$\max_{0 \leq s \leq t} B(s) \geq a \iff T_a \leq t$$

Thus the distribution of for $\max_{0 \leq s \leq t} B(s)$ can be derived via T_a. For $a > 0$

$$\begin{aligned} \mathrm{P}\left(\max_{0 \leq s \leq t} B(s) \geq a\right) &= \mathrm{P}(T_a \leq t) \\ &= 2\mathrm{P}(B(t) \geq a) = \mathrm{P}(|B(t)| \geq a) \\ &= 2 - 2\Phi(a/\sqrt{t}) \end{aligned}$$

Note this implies $\max_{0 \leq s \leq t} B(s)$ have the same distribution as $|B(t)|$.

Stopping Time & Strong Markov Property

For a continuous-time stochastic process $\{X(t), t \geq 0\}$, a *stopping time T with respect to* $\{X(t), t \geq 0\}$ is a nonnegative random variable, such that the event $\{T \leq t\}$ depends only on $\{X(s) : 0 \leq s \leq t\}$, but not $\{X(s) : s > t\}$.

Example

The hitting time $T_a = \min\{t : B(t) = a\}$ is a stopping time since the event $\{T_a \leq t\}$ is identical to the event $\left\{\max_{0 \leq s \leq t} B(s) \geq a\right\}$

Theorem (Strong Markov Property)

Let $\{B(t), t \geq 0\}$ be a standard Brownian Motion, and let T be a stopping time respective to $\{B(t), t \geq 0\}$. Then

(a) Define $Z(t) = B(t + T) - B(T)$, $t \geq 0$. Then $\{Z(t), t \geq 0\}$ is also a standard Brownian Motion

(b) For each $t > 0$, $\{Z(s), 0 \leq s \leq t\}$ is independent of $\{B(u), 0 \leq u \leq T\}$

Reflection Principle

Let T_a be the first passage time to the value a of a standard Brownian Motion $\{B(t), t \geq 0\}$. Define a new process

$$\overline{B}(t) = \begin{cases} B(t) & \text{for } t \leq T_a \\ 2a - B(t) & \text{for } t > T_a \end{cases}$$

Then $\{\overline{B}(t), t \geq 0\}$ is also a standard Brownian Motion.

Reason: For $t > T_a$, note

$$B(t) = a + B(t) - a = B(T_a) + B(t) - B(T_a).$$

- By Strong Markov Property, $B(s + T_a) - B(T_a) = B(s + T_a) - a$ is also a Brownian Motion, independent of $\{B(s), 0 \leq s \leq T_a\}$.
- Also note that if $\{B(t), t \geq 0\}$ is a standard Brownian motion, so is $\{-B(t), t \geq 0\}$. Hence $\{a - B(s + T_a), s \geq 0\}$ is also a Brownian Motion.

$$\begin{aligned} \text{So } \{B(t), t > T_a\} &= \{a + B(t) - a, t > T_a\} \\ &\sim \{a + a - B(t), t > T_a\} = \{2a - B(t), t > T_a\}. \end{aligned}$$

Brownian Motion Absorbed at a Value

Let $\{B(t)\}$ be a Brownian Motion.

For $a > 0$, a Brownian Motion absorbed at a value a is defined as

$$B_a(t) = \begin{cases} B(t) & \text{if } \max_{0 \leq s \leq t} B(s) < a \\ a & \text{if } \max_{0 \leq s \leq t} B(s) \geq a \end{cases}$$

What is the distribution of $B_a(t)$? For $x < a$,

$$\begin{aligned} \mathrm{P}(B_a(t) \leq x) &= \mathrm{P}\Big(B(t) \leq x, \max_{0 \leq s \leq t} B(s) < a\Big) \\ &= \mathrm{P}(B(t) \leq x) - \mathrm{P}\Big(B(t) \leq x, \max_{0 \leq s \leq t} B(s) \geq a\Big) \\ &= \mathrm{P}(B(t) \leq x) - \mathrm{P}(B(t) \leq x, T_a \leq t) \end{aligned}$$

where the last equality comes from the fact

$$\left\{\max_{0 \leq s \leq t} B(s) \geq a\right\} \Leftrightarrow \{T_a \leq t\}.$$

Brownian Motion Absorbed at a Value

By the Reflection principle,

$$\begin{aligned} &\mathrm{P}(B(t) \leq x, T_a \leq t) \\ &= \mathrm{P}(B(t) \geq 2a - x, T_a \leq t) = \mathrm{P}(B(t) \geq 2a - x) \end{aligned}$$

since $x \leq a$, $B(t) \geq 2a - x > a$ implies $T_a \leq t$.

In summary, the CDF of $B_a(t)$ is

$$\begin{aligned} \mathrm{P}(B_a(t) \leq x) &= \mathrm{P}\Big(B(t) \leq x, \max_{0 \leq s \leq t} B(s) < a\Big) \\ &= \mathrm{P}(B(t) \leq x) - \mathrm{P}(B(t) \geq 2a - x) \\ &= \Phi\left(\frac{x}{\sqrt{t}}\right) - 1 + \Phi\left(\frac{2a - x}{\sqrt{t}}\right) \end{aligned}$$