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Section 8.7 The Model G/M/1

8.7 The Model G/M/1

The $G/M/1$ model assumes

- i.i.d times between successive arrivals with an arbitrary distribution G
- i.i.d service times $\sim \text{Exp}(\mu)$
- a single server; and
- first come, first serve

Just like $M/G/1$ system, there is also a discrete-time Markov chain embedded in an $G/M/1$ system. Let

$$\begin{aligned} Y_n &= \#\text{ of customers in the system seen by the } n\text{th arrival}, \quad n \ge 1 \\ D_n &= \#\text{ of customers the server can possibly serve} \\ &\quad \text{between the } (n-1)\text{st and the } n\text{th arrival}, \quad n \ge 1 \end{aligned}$$

Observed that $\{Y_n, n \ge 0\}$ and $\{D_n, n \ge 1\}$ are related as follows

$$Y_{n+1} = \begin{cases} Y_n + 1 - D_{n+1} & \text{if } Y_n + 1 \ge D_{n+1} \\ 0 & \text{if } Y_n + 1 < D_{n+1} \end{cases}, \quad n \ge 1$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

- By the memoryless property of the exponential service time, the remaining service time of the customer being served at an arrival is also $\sim \text{Exp}(\mu)$.
- Thus starting from the $(n-1)$st arrival, the events of completion of servicing a customer constitute a Poisson process of rate μ.
- Let G_n be the time elapsed between the $(n-1)$st and the nth arrival.
- Then given G_n, D_n is Poisson with mean μG_n.
- As G_n's are i.i.d $\sim G$, we can conclude that $D_1, D_2, \ldots$ are i.i.d. with distribution

$$\begin{aligned} \delta_k = \mathrm{P}(D_n = k) &= \int_0^\infty \mathrm{P}(D_n = k | G_n = y) G(dy) \\ &= \int_0^\infty \frac{(\mu y)^k}{k!} e^{-\mu y} G(dy) \end{aligned}$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

The transition probabilities P_{ij} for the Markov chain $\{Y_n, n \ge 0\}$ are thus:

$$\begin{aligned} P_{ij} &= \mathrm{P}(Y_{n+1} = j | Y_n = i) \\ &= \begin{cases} \mathrm{P}(D_{n+1} \ge i+1) = \sum_{k=i+1}^\infty \delta_k & \text{if } j = 0 \\ \mathrm{P}(D_{n+1} = i+1-j) = \delta_{i+1-j}, & \text{if } j \ge 1, i+1 \ge j \\ 0 & \text{if } i+1 < j \end{cases} \end{aligned}$$

i.e., the transition probability matrix is

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\ \begin{pmatrix} \sum_{k=1}^\infty \delta_k & \delta_0 & 0 & 0 & 0 & \cdots \\ \sum_{k=2}^\infty \delta_k & \delta_1 & \delta_0 & 0 & 0 & \cdots \\ \sum_{k=3}^\infty \delta_k & \delta_2 & \delta_1 & \delta_0 & 0 & \cdots \\ \sum_{k=4}^\infty \delta_k & \delta_3 & \delta_2 & \delta_1 & \delta_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array}$$

A Markov Chain embedded in $G/M/1$ (Cont'd)

To find the stationary distribution $\pi_i = \lim_{n\to\infty} \mathrm{P}(Y_n = i)$, $i = 0, 1, 2, \ldots$, we have to solve the equations

$$\pi_j = \sum_{i=0}^\infty \pi_i P_{ij} = \sum_{i=j-1}^\infty \pi_i \delta_{i+1-j}, \; j \ge 1 \quad \text{and} \quad \sum_{j=0}^\infty \pi_j = 1$$

Let us try a solution of the form $\pi_j = c\beta^j$, $j \ge 0$. Substituting into the equation above leads to

$$\begin{aligned} c\beta^j &= \sum\nolimits_{i=j-1}^\infty c\beta^i \delta_{i+1-j} \quad (\text{Divide both sides by } c\beta^{j-1}) \\ \Rightarrow \quad \beta &= \sum\nolimits_{i=j-1}^\infty \beta^{i+1-j} \delta_{i+1-j} = \sum\nolimits_{i=0}^\infty \beta^i \delta_i \end{aligned}$$

Observe that $\sum_{i=0}^\infty \beta^i \delta_i$ is exactly the generating function of D_n $g(s) = \mathbb{E}[s^{D_n}]$ taking value at $s = \beta$.
Thus if we can find $0 < \beta < 1$ such that $\beta = g(\beta)$, then

$$\pi_j = (1-\beta)\beta^j, \quad j \ge 0$$

is a stationary distribution of $\{Y_n\}$.

A Markov Chain embedded in $G/M/1$ (Cont'd)

The equation

$$\beta = g(\beta)$$

has a solution between 0 and 1 iff $g'(1) = E[D_n] = \mu\mathbb{E}[G_n] > 1$ since

This condition is intuitive since if

$$\begin{aligned} &\text{the average service time } 1/\mu \\ &> \text{ the average interarrival time of customers } \mathbb{E}[G_n], \end{aligned}$$

the queue will become longer and longer and the system will ultimately explode.

PASTA Principle Does Not Apply to G/M/1

With the stationary distribution $\{\pi_j, j \geq 0\}$, one might attempt to calculate L, the average number of customers in the system as

$$\mathbb{E}[Y_n] = \sum_{k=0}^{\infty} \pi_k = \sum_{k=0}^{\infty} k(1-\beta)\beta^k = \frac{\beta}{1-\beta}.$$

However, the PASTA principle does not apply as the arrival process is not Poisson. Recall

$$a_k = \pi_k = \text{proportion of arrivals see } k \text{ in the system}$$
$$P_k = \text{proportion of time having } k \text{ customers in the system,}$$

Observe that $\pi_k \neq P_k$ since the longer the interarrival time G_n, the larger D_n, and hence the smaller Y_n. Hence

$$L = \sum_{k=0}^{\infty} P_k \neq \mathbb{E}[Y_n] = \sum_{k=0}^{\infty} \pi_k.$$

W of G/M/1

Though we cannot use $\{\pi_j\}$ to find L, we can use it to find W. Let W_n be the waiting time of nth customer in the system. If he/she see k customers at arrival, then W_n is the total service time of $k+1$ customers. That is,

$$\mathbb{E}[W_n | Y_n = k] = \mathbb{E}[\text{sum of } k+1 \text{ i.i.d. Exp}(\mu) \text{ service times}] = \frac{k+1}{\mu}.$$

Thus

$$W = \sum_{k=0}^{\infty} \mathbb{E}[W_n | Y_n = k] \mathrm{P}(Y_n = k) = \sum_{k=0}^{\infty} \mathbb{E}[W_n | Y_n = k] \pi_k = \sum_{k=0}^{\infty} \frac{k+1}{\mu}(1-\beta)\beta^k = \frac{1}{\mu(1-\beta)}$$

Here we use the identity $\sum_{k=0}^{\infty}(k+1)x^k = \frac{1}{(1-x)^2}$.

L, W_Q, L_Q of G/M/1

By the Little's Formula, we know $L = \lambda W$, in which λ is the arrival rate of customers, which is the reciprocal of the mean interarrival time $\mathbb{E}[G_n]$

$$\lambda = \frac{1}{\mathbb{E}[G_n]}$$

Thus

$$L = \lambda W = \frac{1}{\mathbb{E}[G_n]} \frac{1}{\mu(1-\beta)} = \frac{1}{\mu \mathbb{E}[G_n](1-\beta)}$$

Moreover,

$$W_Q = W - \mathbb{E}[\text{Service Time}] = W - \frac{1}{\mu} = \frac{\beta}{\mu(1-\beta)}$$
$$L_Q = \lambda W_Q = \frac{\beta}{\mu \mathbb{E}[G_n](1-\beta)}$$

8.9.3 $G/M/k$

Just like $G/M/1$ system, $G/M/1$ system can also be analyzed as a Markov Chain. Let

$$Y_n = \# \text{ of customers in the system seen by the } n\text{th arrival}, \ n \geq 1$$
$$D_n = \# \text{ of customers the } k \text{ servers can possibly serve between the } (n-1)\text{st and the } n\text{th arrival}, \ n \geq 1$$

Observed again that $\{Y_n, n \geq 0\}$ and $\{D_n, n \geq 1\}$ are related as follows

$$Y_{n+1} = \begin{cases} Y_n + 1 - D_{n+1} & \text{if } Y_n + 1 \geq D_{n+1} \\ 0 & \text{if } Y_n + 1 < D_{n+1} \end{cases}, \quad n \geq 1$$

One can show that the distribution of D_{n+1} depends on Y_n but not $Y_{n-1}, Y_{n-2}, \ldots$ and hence $\{Y_n\}$ is a Markov chain. The transition probabilities are given in [IPM10e] p.565-566.

8.9.4 $M/G/k$

Unlike $G/M/k$, the method to analyze $M/G/1$ cannot be used to analyze $M/G/k$. If we follow the lines as we do in $M/G/1$

$$Y_n = \# \text{ of customers in the system leaving behind at the } n\text{th departure}, \ n \geq 1$$
$$D_n = \# \text{ of customers entered the system during the service time of the } n\text{th customer}, \ n \geq 1$$

As there are more than one server, the service times are not disjoint, and hence D_n's are not independent.

In fact, there is NO known exact formula for L, W, L_Q, W_Q of an $M/G/k$ system.

10.6 Gaussian Processes

Definition 10.2

A stochastic process $\{X(t), t \geq 0\}$ is called a *Gaussian process* if $X(t_1), \ldots, X(t_n)$ has a multivariate normal distribution for all $t_1, \ldots, t_n$.

Because a multivariate normal distribution is completely determined by the marginal mean values and the covariance values it follows that the properties of a Gaussian process is completely determined by its *mean function*

$$m(t) = \mathbb{E}[X(t)]$$

and *covariance function*

$$C(s,t) = \mathrm{Cov}(X(s), X(t)).$$

That is, two Gaussian processes are the same if

their **mean functions** and **covariance functions** are identical.

Definition of a Brownian Motion

Definition 1 A stochastic process $\{B(t), t \geq 0\}$ is said to be a if

(i) $B(0) = 0$;

(ii) $\{B(t), t \geq 0\}$ has stationary and independent increments;

(iii) for every $t, s > 0$, $B(t+s) - B(s) \sim N(0, \sigma^2 t)$

A Brownian motion with $\sigma = 1$ is called a *standard Brownian motion process*

In fact, we can show that, as a function of t, the path of $B(t)$ is **continuous** w/ prob. 1.

Covariance Function of a Brownian Motion

For $t > s$

$$\begin{aligned}\mathrm{Cov}[B(t), B(s)] &= \mathrm{Cov}[B(t) - B(s) + B(s), B(s)] \\ &= \mathrm{Cov}[B(t) - B(s), B(s)] + \mathrm{Cov}[B(s), B(s)] \\ &= 0 + \mathrm{Var}[B(s)] \quad \text{(by indep. increment)} \\ &= \sigma^2 s\end{aligned}$$

Thus

$$\mathrm{Cov}(B(t), B(s)) = \sigma^2 \min(s, t)$$

$\{B(t), t \geq 0\}$ is a Brownian motion.

Alternatively, a Brownian motion can be defined as a Gaussian process with mean function

$$m(t) = \mathbb{E}[B(t)] = 0$$

and covariance function

$$C(s, t) = \mathrm{Cov}(B(s), B(t)) = \sigma^2 \min(s, t).$$

Properties of a Brownian Motion

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion, then each of the following process is also a standard Brownian motion:

(i) $\{-B(t), t \geq 0\}$ (ii) $\{B(t+s) - B(s), t \geq 0\}$
(iii) $\{aB(t/a^2), t \geq 0\}$ (iv) $\{tB(1/t), t \geq 0\}$

Pf. We'll prove (iv) only. The proofs for the rest are similar. Clearly $\{tB(1/t), t \geq 0\}$ is a Gaussian process since it is a linear function of a Brownian motion process.

$$\begin{aligned}\mathbb{E}[tB(1/t)] &= t\mathbb{E}[B(1/t)] = 0 \quad \text{since } B(1/t) \sim N(0, 1/t) \\ \mathrm{Cov}[tB(1/t), sB(1/s)] &= ts\mathrm{Cov}[B(1/t), B(1/s)] \\ &= ts \min\left(\frac{1}{t}, \frac{1}{s}\right) = \begin{cases} ts(1/t) = s & \text{if } t > s \\ ts(1/s) = t & \text{if } t \leq s \end{cases} \\ &= \min(s, t)\end{aligned}$$

As the Gaussian process $\{tB(1/t), t \geq 0\}$ has the same mean function and variance function as a standard Brownian motion, it is also a standard Brownian motion.