STAT253/317 Winter 2014 Lecture 1

Yibi Huang

January 8, 2013

4.1 Introduction to Markov Chains

Stochastic Processes

A stochastic process is a family of random variables $\{X_t : t \in \mathcal{T}\}$ such that

- For each $t \in \mathcal{T}$, X_t is a random variable
- The index set $\mathcal{T}$ can be discrete or continuous
  - $\mathcal{T} = \{0, 1, 2, 3, 4\}$
  - $\mathcal{T} = \mathbb{R}, \mathbb{R}^+, \mathbb{R}^2, \mathbb{R}^3$

Examples:

- Discrete Time Markov Chains .... Chapter 4
- Poisson Processes, Counting Processes .... Chapter 5
- Continuous Time Markov Chain .... Chapter 6
- Renewal Theory .... Chapter 7
- Queuing Theory .... Chapter 8
- Brownian Motion .... Chapter 10

4.1 Introduction to Markov Chain

Consider a stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ taking values in a finite or countable set $\mathfrak{X}$.

- $\mathfrak{X}$ is called the **state space**
- If $X_n = i$, $i \in \mathfrak{X}$, we say the process is in state i at time n
- Since $\mathfrak{X}$ is countable, there is a 1-1 map from $\mathfrak{X}$ to the set of non-negative integers $\{0, 1, 2, 3, \ldots\}$
  From now on, we assume $\mathfrak{X} = \{0, 1, 2, 3, \ldots\}$

Definition

A stochastic process $\{X_n : n = 0, 1, 2, \ldots\}$ is called a **Markov chain** if it has the following property:

$$\begin{aligned}&P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_2 = i_2, X_1 = i_1, X_0 = i_0)\\ &= P(X_{n+1} = j | X_n = i)\end{aligned}$$

for all states i_0, i_1, $i_2, \ldots, i_{n-1}$, i, $j \in \mathfrak{X}$ and $n \geq 0$.

Transition Probability Matrix

If $P(X_{n+1} = j | X_n = i) = P_{ij}$ does not depend on n, then the process $\{X_n : n = 0, 1, 2, \ldots\}$ is called a **stationary Markov chain**. From now on, we consider stationary Markov chain only.

$\{P_{ij}\}$ is called the **transition probabilities**. The matrix

$$\mathbb{P} = \begin{pmatrix} P_{00} & P_{01} & P_{02} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ P_{i0} & P_{i1} & P_{i2} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{pmatrix}$$

is called the **transition probability matrix**.

Naturally, the transition probabilities $\{P_{ij}\}$ satisfies the following

$$1) P_{ij} \geq 0 \text{ for all } i, j \quad \text{and} \quad 2) \sum_j P_{ij} = 1 \text{ for all } i.$$

In other words,

$$\mathbb{P}\mathbf{1} = \mathbf{1}, \quad \text{where } \mathbf{1} = (1, 1, \cdots, 1, \cdots)^T$$

Example 1: Two-State Markov Chain

Consider the state of a phone

$$X_n = \begin{cases} 0 & \text{if the phone is \textbf{free} at time interval } n \\ 1 & \text{if the phone is \textbf{busy} at time interval } n. \end{cases}$$

- For each time interval, $\mathrm{P}(\text{one call comes in that interval}) = \alpha$ (assume at most one call per interval).
- If the phone is busy, incoming calls do not get through.
- If the phone is busy during a time interval, there is a probability β that it will be free during the next interval.

This gives a Markov chain with state space

$$\mathfrak{X} = \{\text{free}, \text{busy}\} = \{0, 1\}$$

and transition matrix

$$\mathbb{P} = \begin{matrix} \\ \text{free} \\ \text{busy} \end{matrix} \begin{matrix} \begin{matrix} \text{free} & \text{busy} \end{matrix} \\ \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix} \end{matrix}$$

Example 2: Random Walk

$$\mathfrak{X} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\} = \mathbb{Z} = \text{all integers}$$

$$P_{ij} = \begin{cases} p & \text{if } j = i + 1 \\ 1 - p & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases}$$

Example 2: Random Walk (Cont'd)

$$
\mathbb{P} = \begin{array}{c} \\ \vdots \\ -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} \cdots & -3 & -2 & -1 & 0 & 1 & 2 & 3 & \cdots \end{array} \\
\left(\begin{array}{ccccccccc}
\ddots & \ddots & & & & & & & \\
\ddots & 0 & p & & & & & & \\
& 1-p & 0 & p & & & & & \\
& & 1-p & 0 & p & & & & \\
& & & 1-p & 0 & p & & & \\
& & & & 1-p & 0 & p & & \\
& & & & & 1-p & 0 & p & \\
& & & & & & 1-p & 0 & \ddots \\
& & & & & & & \ddots & \ddots
\end{array}\right)
\end{array}
$$

Example 3: Ehrenfest Diffusion Model

Two containers A and B, containing a sum of $2a$ balls. At each stage, a ball is selected at random from the totality of $2a$ balls, and move to the other container. Let

$$X_0 = \# \text{ of balls in container } A \text{ in the beginning}$$
$$X_n = \# \text{ of balls in container } A \text{ after } n \text{ movements}, \ n = 1, 2, 3, \ldots$$

$$\mathfrak{X} = \{0, 1, 2, \ldots, 2a\}$$

$$
P_{ij} = \begin{cases} \dfrac{i}{2a} & \text{if } j = i - 1 \\ \dfrac{2a - i}{2a} & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}
$$

Example 4: Discrete Queuing Process

A line of customers await in front of 1 server.

- It takes one unit of time to serve 1 customer
- During each period of time, only 1 customer is served.
- If no customer awaits, the server idles

Let $\xi_n = \#$ of customers arriving in the n-th period. Suppose $\{\xi_n, n = 0, 1, 2, \cdots\}$ are i.i.d. with

$$\mathrm{P}(\xi_n = k) = a_k, \quad k = 0, 1, 2, \ldots$$
$$a_k \geq 0 \text{ for all } k, \quad \text{and } \sum_{k=0}^{\infty} a_k = 1$$

Let $X_n = \#$ of customers awaits during the n-th period, including the one being served. Then

$$
X_{n+1} = \begin{cases} X_n - 1 + \xi_n & \text{if } X_n \geq 1 \\ \xi_n & \text{if } X_n = 0 \end{cases}
$$

Example 3: Discrete Queuing Process (Cont'd)

The transition probability matrix will be

$$
\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & \cdots \end{array} \\
\left(\begin{array}{cccccc}
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\
0 & a_0 & a_1 & a_2 & a_3 & \cdots \\
0 & 0 & a_0 & a_1 & a_2 & \cdots \\
0 & 0 & 0 & a_0 & a_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right)
\end{array}
$$