7.4 Renewal Reward Processes

7.5.1 Alternating Renewal Processes

Proof of Proposition 7.3(a)

We give the proof for (a) only. To prove this, write

\[ R(t) = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \times \frac{N(t)}{t} \]

By the strong law of large numbers we obtain

\[ \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \rightarrow E[R_1] \quad \text{as } t \to \infty \]

and by Proposition 7.1

\[ \frac{N(t)}{t} \rightarrow \frac{1}{E[X_1]} \quad \text{as } t \to \infty \]

The result thus follows.

Example 7.12 (A Car Buying Model) Solution

- An event occurs whenever Mr. Brown buys a new car
- Interarrival times: \( X_i = \min(Y_i, T) \)
- Cost incurred in the \( i \)th cycle: \( R_i = C_1 + C_2 1_{(Y_i \leq T)} \)
- Are \( (X_i, R_i) \), \( i = 1, 2, \ldots \) i.i.d?

\[ E[X_i] = \int_0^\infty \min(y, T)h(y)dy = \int_0^T yh(y)dy + T(1 - H(T)) \]

\[ E[R_i] = C_1 + C_2 P(Y_i \leq T) = C_1 + C_2 H(T) \]

- long-run average cost

\[ = \frac{C_1 + C_2 H(T)}{\int_0^T yh(y)dy + T(1 - H(T))} \]

7.4 Renewal Reward Processes

Let \( \{N(t), t \geq 0\} \) be a renewal process with i.i.d. interarrival times \( X_i, i = 1, 2, \ldots \). Let \( R_i \), \( i = 1, 2, \ldots \) be i.i.d random variables. \( R_i \) may depend on the \( i \)th interarrival time \( X_i \), but \( (X_i, R_i) \) are i.i.d random variable pairs. The compound process

\[ R(t) = \sum_{j=1}^{N(t)} R_j \]

is called a renewal reward process. \( R(t) \) may be considered as reward earned during the \( i \)th cycle, and \( R(t) \) represents the total reward earned up to time \( t \).

**Proposition 7.3** If \( E[R_1] < \infty \) and \( E[X_1] < \infty \), then

(a) with probability 1, \( \lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[R_1]}{E[X_1]} \)

(b) \( \lim_{t \to \infty} \frac{E[R(t)]}{t} = \frac{E[R_1]}{E[X_1]} \)

Example 7.16 & 7.17

Let \( \{N(t), t \geq 0\} \) be a renewal process with i.i.d. interarrival times \( X_i, i = 1, 2, \ldots \). Consider the current age of the item in use at time \( t \)

\[ A(t) = t - S_{N(t)} \]

What is the long-run average of age

\[ \lim_{t \to \infty} \frac{\int_{0}^{t} A(s)ds}{t} \]

Also consider the residual life of the item in use at time \( t \)

\[ Y(t) = S_{N(t)} + 1 - t \]

What is the long-run average of residual life

\[ \lim_{t \to \infty} \frac{\int_{0}^{t} Y(s)ds}{t} \]
Solution to Example 7.16

Let’s try to turn $\int_0^t A(s)ds$ into a renewal reward process:

Note $\int_0^{N(t)} A(s)ds < \int_0^t A(s)ds < \int_0^{N(t)+1} A(s)ds$, and

$$\int_0^{N(t)} A(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} A(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} s - S_{i-1}ds$$

$$= \sum_{i=1}^{N(t)} \int_0^{S_i} udu \quad \text{(let } u = s - S_{i-1})$$

$$= \sum_{i=1}^{N(t)} \frac{S_i^2}{2} = R(t),$$

where $R(t) = \sum_{i=1}^{N(t)} R_i$ is a renewal reward process with $R_0 = \frac{X_0^2}{2}$. Similarly, one can show that

$$\int_0^{N(t)+1} A(s)ds = \sum_{i=1}^{N(t)+1} R_i = R(t) + R_{N(t)+1}.$$


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Solution to Example 7.17

Similarly, from that

$$\int_0^{N(t)} Y(s)ds \leq \int_0^t Y(s)ds \leq \int_0^{N(t)+1} Y(s)ds,$$

one can show that

$$\int_0^{N(t)} Y(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} (S_i - s)ds = \sum_{i=1}^{N(t)} \int_0^{S_i} udu \quad \text{(let } u = s - S_{i-1})$$

$$= \sum_{i=1}^{N(t)} \frac{X_i^2}{2} = R(t),$$

and that $\int_0^{N(t)+1} Y(s)ds = \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2} = R(t) + R_{N(t)+1}$. By the same argument, the long-run average of residual life of the item in use is

$$\lim_{t \to \infty} \frac{\int_0^t Y(s)ds}{t} = \lim_{t \to \infty} R(t) = \frac{E[R_1]}{E[X_1]} = \frac{E[X_1]^2}{2E[X_1]}$$

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7.5.1. Alternating Renewal Processes (Cont’d)

The alternating renewal process can be regarded as a reward process with reward $R_i = Z_i$,

$$R(t) = \sum_{i=1}^{N(t)} Z_i$$

Then

$$R(t) \leq \int_0^t U(s)ds \leq R(t) + Z_{N(t)+1}$$

By Proposition 7.3, with probability 1,

$$\lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[Z_1]}{E[X_1]} = \frac{E[Z_1]}{E[Z_1] + E[Y_1]}$$

and hence

$$\lim_{t \to \infty} \frac{\int_0^t U(s)ds}{t} = \lim_{t \to \infty} \frac{R(t)}{t} = \frac{E[Z_1]}{E[Z_1] + E[Y_1]} = \frac{E[ON]}{E[ON] + E[OFF]}$$

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Definition: A Lattice Distribution

A random variable $X$ is said to have a lattice distribution if there is an $h > 0$ for which

$$\sum_{k=-\infty}^{\infty} P(X = kh) = 1$$

in which the largest $h$ is called the span of $X$.

Example 1. Many discrete distributions, like Poisson, Binomial, are lattice distributions.

Example 2. Continuous distributions are non-lattice. Mixtures of discrete and continuous distributions are also non-lattice.

Remark: If $X_i$‘s are i.i.d. with a common lattice distribution, then

$$S_n = X_1 + \ldots + X_n$$

also has a lattice distribution for all $n$.

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**Theorem:** If the distribution of the interarrival times is non-lattice, then

\[
\lim_{t \to \infty} P(\text{the system is on at time } t) = \lim_{t \to \infty} P(U(t) = 1) = \frac{\mathbb{E}[Z]}{\mathbb{E}[Z] + \mathbb{E}[Y]}
\]

**Exercise 7.39**

- Two independent machines, each functions for an exponential time with rate \( \lambda \)
- A single repairman. All repair times are independent with distribution function \( G \)
- If the repairmen is free when a machine fails, he will begin repairing that machine immediately; Otherwise, then that machine must wait until the other machine has been repaired.
- Once repaired, a machine is as good as new.
- What proportion of time is the repairmen idle?

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**Example 7.23 & 7.24**

Let \( \{N(t), t \geq 0\} \) be a renewal process with i.i.d. interarrival times \( X_i, i = 1, 2, \ldots \), where \( \mu = \mathbb{E}[X_1] \) and \( F(x) = P(X_i \leq x) \).

Consider the current age of the item in use at time \( t \)

\[ A(t) = t - S_{N(t)} \]

and the residual life of the item in use at time \( t \)

\[ Y(t) = S_{N(t)+1} - t \]

**Proposition** The long-run proportion of time that \( A(t) \leq x \) is the same as the long-run proportion of time that \( Y(t) \leq x \), and is

\[ F_e(x) = \frac{1}{\mu} \int_0^x (1 - F(u))du \]

Furthermore, if \( F \) is non-lattice,

\[ \lim_{t \to \infty} P(A(t) \leq x) = \lim_{t \to \infty} P(Y(t) \leq x) = F_e(x) \]

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**Example 7.24 (Con’d)**

For \( Y(t) \),

- let’s say the system is OFF at time \( t \) if \( Y(t) \leq x \)
- length of OFF time \( Z_i = \min(X_i, x) \)

\[ \mathbb{E}[Z_i] = \mathbb{E}[\min(X_i, x)] = \int_0^x (1 - F(u))du \]

- length of a cycle = \( X_i \), \( \mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu \)
- long-run proportion of time that \( Y(t) \leq x \)

\[ \frac{\mathbb{E}[\text{OFF}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^x (1 - F(u))du \]

**Remark 1:** The ON time in Example 7.23 is not the same as the ON time in Example 7.24

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**Exercise 7.39 Solutions**

- The system is ON when the repairmen is idling, OFF when busy
- length of ON time: \( Z \sim \text{Exp}(2\lambda), \mathbb{E}[Z] = 1/(2\lambda) \)
- length of OFF time \( Y, \mathbb{E}[Y] = ? \)
- \( T = \) the time it takes to repair the first failing machine. \( T \sim G \)
- \( U = \) the time the working machine can function after the first machine failed. By the memoryless property, \( U \sim \text{Exp}(\lambda) \)
- Note that \( Y = T + Y'1_{\{T > U\}} \) where \( Y' \) is the time the repairmen remains busy after the first failing machine is fixed. Note that \( Y' \) is independent of \( T \) and \( U \), and has the same distribution as \( Y \). Thus

\[ \mathbb{E}[Y] = \mathbb{E}[T] + \mathbb{E}[Y']P(T > U) = \mathbb{E}[Y] = \frac{\mathbb{E}[T]}{P(T < U)} \]

- long-run proportion of ON time =

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**Example 7.23 (Con’d)**

For \( A(t) \),

- let’s say the system is ON at time \( t \) if \( A(t) \leq x \)
- length of ON time \( Y_t = \min(X_t, x) \)

\[ \mathbb{E}[Y_t] = \mathbb{E}[\min(X_t, x)] = \int_0^\infty P(\min(X_t, x) > u)du \]

\[ = \int_0^\infty (1 - F(u))du \]

- length of a cycle = \( X_t \), \( \mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_t] = \mu \)
- long-run proportion of time that \( A(t) \leq x \)

\[ \frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1}{\mu} \int_0^\infty (1 - F(u))du \]

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**About \( F_e \)**

The density and \( k \)th moment of the distribution \( F_e \) is

\[ f_e(x) = \frac{1}{\mu} (1 - F(x)) \]

\[ \int_0^\infty x^k f_e(x)dx = \frac{\mathbb{E}[X^{k+1}]}{(k+1)\mathbb{E}[X]} \]

Recall that

\[ m(t) = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{\mu t} \]

If \( F \) is non-lattice, since the limiting distribution of \( Y(t) \) is \( F_e \), we have

\[ \lim_{t \to \infty} \mathbb{E}[Y(t)] = \frac{\mu^2 + \sigma^2}{2\mu} \]

Thus

\[ m(t) = \frac{t}{\mu} - 1 + \frac{\mu^2 + \sigma^2}{2\mu^2} + o(t) \]

\[ = \frac{t}{\mu} + \frac{\sigma^2 - \mu^2}{2\mu^2} + o(t) \]

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