STAT253/317 Winter 2013 Lecture 17

February 18, 2013

7.4 Renewal Reward Processes
7.5.1 Alternating Renewal Processes

7.4 Renewal Reward Processes

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times X_i, $i = 1, 2, \ldots$. Let R_i $i = 1, 2, \ldots$ be i.i.d random variables. R_i may depend on the ith interarrival time X_i, but (X_i, R_i) are i.i.d. random variable pairs. The compound process

$$R(t) = \sum_{i=1}^{N(t)} R_i$$

is called a *renewal reward process*. R_i may be considered as reward earned during the ith cycle, and $R(t)$ represents the total reward earned up to time t.

Proposition 7.3 If $\mathbb{E}[R_1] < \infty$ and $\mathbb{E}[X_1] < \infty$, then

(a) with probability 1, $\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$

(b) $\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]}$

Proof of Proposition 7.3(a)

We give the proof for (a) only. To prove this, write

$$\frac{R(t)}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{t} = \frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \times \frac{N(t)}{t}$$

By the strong law of large numbers we obtain

$$\frac{\sum_{n=1}^{N(t)} R_n}{N(t)} \to \mathbb{E}[R_1] \quad \text{as } t \to \infty$$

and by Proposition 7.1

$$\frac{N(t)}{t} \to \frac{1}{\mathbb{E}[X_1]} \quad \text{as } t \to \infty$$

The result thus follows.

Example 7.12 (A Car Buying Model)

- Mr. Brown buy a new car whenever his old one breaks down or reaches the age of T years
- Let Y_i be the lifetime of his ith car. Suppose Y_i's are i.i.d with CDF

$$H(y) = \mathrm{P}(Y \leq y), \quad \text{and density } h(y) = H'(y).$$

- Cost to by a new car $= C_1$;
- If the car breaks down, an additional cost of C_2 is incurred.
- What's Mr. Brown's long run average cost?

Example 7.12 (A Car Buying Model) Solution

- An event occurs whenever Mr. Brown buys a new car
- Interarrival times: $X_i = \min(Y_i, T)$
- Cost incurred in the ith cycle: $R_i = C_1 + C_2\mathbf{1}_{\{Y_i \leq T\}}$
- Are (X_i, R_i), $i = 1, 2, \ldots$ i.i.d?
- $$\mathbb{E}[X_i] = \int_0^\infty \min(y, T)h(y)dy = \int_0^T yh(y)dy + T(1 - H(T))$$
- $$\mathbb{E}[R_i] = C_1 + C_2\mathrm{P}(Y_i \leq T) = C_1 + C_2H(T)$$
- long-run average cost

$$= \frac{C_1 + C_2H(T)}{\int_0^T yh(y)dy + T(1 - H(T))}$$

Example 7.16 & 7.17

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times X_i, $i = 1, 2, \ldots$. Consider the current age of the item in use at time t

$$A(t) = t - S_{N(t)}.$$

What is the long-run average of age

$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t}?$$

Also consider the residual life of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

What is the long-run average of residual life

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)ds}{t}?$$

Solution to Example 7.16

Let's try to turn $\int_0^t A(s)ds$ into a renewal reward process:
Note $\int_0^{S_{N(t)}} A(s)ds \le \int_0^t A(s)ds < \int_0^{S_{N(t)+1}} A(s)ds$, and

$$\int_0^{S_{N(t)}} A(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} A(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} s - S_{i-1} ds$$
$$= \sum_{i=1}^{N(t)} \int_0^{X_i} udu \quad (\text{let } u = s - S_{i-1})$$
$$= \sum_{i=1}^{N(t)} \frac{X_i^2}{2} = R(t),$$

where $R(t) = \sum_{i=1}^{N(t)} R_i$ is a renewal reward process with $R_i = X_i^2/2$. Similarly, one can show that

$$\int_0^{S_{N(t)+1}} A(s)ds = \sum_{i=1}^{N(t)+1} R_i = R(t) + R_{N(t)+1}.$$

Solution to Example 7.16 (Cont'd)

Since

$$R(t) \le \int_0^t A(s)ds < R(t) + R_{N(t)+1},$$

and

$$\frac{R_{N(t)+1}}{t} = \frac{X_{N(t)+1}^2}{2t} \to 0 \quad \text{as } t \to \infty,$$

by Proposition 7.3, the long-run average age of the item in use is

$$\lim_{t\to\infty} \frac{\int_0^t A(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}$$

Solution to Example 7.17

Similarly, from that

$$\int_0^{S_{N(t)}} Y(s)ds \le \int_0^t Y(s)ds < \int_0^{S_{N(t)+1}} Y(s)ds,$$

one can show that

$$\int_0^{S_{N(t)}} Y(s)ds = \sum_{i=1}^{N(t)} \int_{S_{i-1}}^{S_i} (S_i - s)ds = \sum_{i=1}^{N(t)} \int_0^{X_i} udu \quad (\text{let } u = S_i - s)$$
$$= \sum_{i=1}^{N(t)} \frac{X_i^2}{2} = R(t)$$

and that $\int_0^{S_{N(t)+1}} Y(s)ds = \sum_{i=1}^{N(t)+1} \frac{X_i^2}{2} = R(t) + R_{N(t)+1}$. By the same argument, the long-run average of residual life of the item in use is

$$\lim_{t\to\infty} \frac{\int_0^t Y(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[X_1^2]}{2\mathbb{E}[X_1]}$$

7.5.1. Alternating Renewal Processes

Considers a system that can be in one of two states: ON or OFF. Initially it is ON, and remains ON for a time Z_1; it then goes OFF and remains OFF for a time Y_1. It then goes ON for a time Z_2; then OFF for a time Y_2; then on, and so on. Suppose (Z_k, Y_k) are i.i.d random vectors, though Z_k and Y_k might depend on each other, and we assume Y_k, Z_k are non-negative with finite means. Then a renewal process $\{N(t), t \ge 0\}$ with interarrival times

$$X_k = Z_k + Y_k, \quad k \ge 1$$

is called an *alternating renewal process.*
Let

$$U(t) = \begin{cases} 1 & \text{if the system is ON at time } t \\ 0 & \text{otherwise} \end{cases}$$

Q1: What is the long-run proportion of time that the system is on?

$$\lim_{t\to\infty} \frac{\int_0^t U(s)ds}{t}?$$

7.5.1. Alternating Renewal Processes (Cont'd)

The alternating renewal process can be regarded as a reward process with reward $R_i = Z_i$,

$$R(t) = \sum_{i=1}^{N(t)} Z_i$$

Then

$$R(t) \le \int_0^t U(s)ds < R(t) + Z_{N(t)+1}$$

By Proposition 7.3, with probability 1,

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[X_1]} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]}$$

and hence

$$\lim_{t\to\infty} \frac{\int_0^t U(s)ds}{t} = \lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1] + \mathbb{E}[Y_1]} = \frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]}$$

Definition: A Lattice Distribution

A random variable X is said to have a *lattice distribution* if there is an $h > 0$ for which

$$\sum_{k=-\infty}^{\infty} \mathrm{P}(X = kh) = 1$$

in which the largest h is called the *span* of X
Example 1. Many discrete distributions, like Poisson, Binomial, are lattice distributions.
Example 2. Continuous distributions are non-lattice. Mixtures of discrete and continuous distributions are also non-lattice.
Remark: If X_i's are i.i.d with a common lattice distribution, then

$$S_n = X_1 + \ldots + X_n$$

also has a lattice distribution for all n

Theorem: If the distribution of the interarrival times is non-lattice, then

$$\lim_{t\to\infty} \mathrm{P}(\text{the system is on at time } t) = \lim_{t\to\infty} \mathrm{P}(U(t)=1) = \frac{\mathbb{E}[Z_1]}{\mathbb{E}[Z_1]+\mathbb{E}[Y_1]}$$

Exercise 7.39

- Two independent machines, each functions for an exponential time with rate λ
- A single repairmen. All repair times are independent with distribution function G
- If the repairmen is free when a machine fails, he will begin repairing that machine immediately; Otherwise, then that machine must wait until the other machine has been repaired.
- Once repaired, a machine is as good as new.
- What proportion of time is the repairmen idle?

Exercise 7.39 Solutions

- The system is ON when the repairmen is idling, OFF when busy
- length of ON time: $Z \sim Exp(2\lambda)$, $\mathbb{E}[Z] = 1/(2\lambda)$
- length of OFF time Y, $\mathbb{E}[Y] = ?$
- $T =$ the time it takes to repair the first failing machine. $T \sim G$
- $U =$ the time the working machine can function after the first machine failed. By the memoryless property, $U \sim Exp(\lambda)$
- Note that

  $$Y = T + Y'\mathbf{1}_{\{T>U\}}$$

  where Y' is the time the repairmen remains busy after the first failing machine is fixed. Note that Y' is independent of T and U, and has the same distribution as Y. Thus

  $$\mathbb{E}[Y] = \mathbb{E}[T] + \mathbb{E}[Y]\mathrm{P}(T>U) \ \Rightarrow\ \mathbb{E}[Y] = \frac{\mathbb{E}[T]}{\mathrm{P}(T<U)}$$

- long-run proportion of ON time =

Example 7.23 & 7.24

Let $\{N(t), t \geq 0\}$ be a renewal process with i.i.d. interarrival times X_i, $i = 1, 2, \ldots$, where $\mu = \mathbb{E}[X_i]$ and $F(x) = \mathrm{P}(X_i \leq x)$. Consider the current age of the item in use at time t

$$A(t) = t - S_{N(t)},$$

and the residual life of the item in use at time t

$$Y(t) = S_{N(t)+1} - t.$$

Proposition The long-run proportion of time that $A(t) \leq x$ is the same as the long-run proportion of time that $Y(t) \leq x$, and is

$$F_e(x) = \frac{1}{\mu}\int_0^x (1-F(u))du.$$

Furthermore, if F is non-lattice,

$$\lim_{t\to\infty} \mathrm{P}(A(t) \leq x) = \lim_{t\to\infty} \mathrm{P}(Y(t)\leq x) = F_e(x)$$

Example 7.23 (Con'd)

For $A(t)$,

- let's say the system is ON at time t if $A(t) \leq x$
- length of ON time $Y_i = \min(X_i, x)$

  $$\mathbb{E}[Y_i] = \mathbb{E}[\min(X_i,x)] = \int_0^\infty \mathrm{P}(\min(X_i,x) > u)du$$
  $$= \int_0^x (1-F(u))du$$

- length of a cycle $= X_i$, $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- long-run proportion of time that $A(t) \leq x$

  $$\frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}]+\mathbb{E}[\text{OFF}]} = \frac{1}{\mu}\int_0^x (1-F(u))du$$

Example 7.24 (Con'd)

For $Y(t)$,

- let's say the system is OFF at time t if $Y(t) \leq x$
- length of OFF time $Z_i = \min(X_i, x)$

  $$\mathbb{E}[Z_i] = \mathbb{E}[\min(X_i,x)] = \int_0^x (1-F(u))du$$

- length of a cycle $= X_i$, $\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}] = \mathbb{E}[X_i] = \mu$
- long-run proportion of time that $Y(t) \leq x$

  $$\frac{\mathbb{E}[\text{OFF}]}{\mathbb{E}[\text{ON}]+\mathbb{E}[\text{OFF}]} = \frac{1}{\mu}\int_0^x (1-F(u))du$$

Remark 1: The ON time in Example 7.23 is not the same as the ON time in Example 7.24

About F_e

The density and kth moment of the distribution F_e is

$$f_e(x) = \frac{1}{\mu}(1-F(x)), \quad \text{and} \quad \int_0^\infty x^k f_e(x)dx = \frac{\mathbb{E}[X^{k+1}]}{(k+1)\mathbb{E}[X]}$$

Recall that

$$\frac{m(t)}{t} = \frac{1}{\mu} - \frac{1}{t} + \frac{\mathbb{E}[Y(t)]}{t\mu}.$$

If F is non-lattice, since the limiting distribution of $Y(t)$ is F_e, we have

$$\lim_{t\to\infty}\mathbb{E}[Y(t)] = \frac{\mu^2+\sigma^2}{2\mu}$$

Thus

$$m(t) = \frac{t}{\mu} - 1 + \frac{\mu^2+\sigma^2}{2\mu^2} + o(t)$$
$$= \frac{t}{\mu} + \frac{\sigma^2-\mu^2}{2\mu^2} + o(t)$$