

# STAT253/317 Winter 2014 Lecture 16

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February 17, 2014

7.3 Limit Theorems

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## Revision of Example 7.7

A coin with probability  $p$  to land heads (and  $q = 1 - p$  to land tails) is tossed continually.

- ▶ What is the probability to get  $k$  **heads in a row** before getting  $k$  **tails in a row**?
- ▶ How many tosses is expected to get  $k$  heads in a row or  $k$  tails in a row?

### Solution.

- ▶ Suppose the coin is tossed at every integer time points  $t = 1, 2, 3, \dots$
- ▶ An *event* occurs whenever getting  $k$  heads in a row. Here we required the rows of heads for different events must be non-overlapping.
- ▶ Define  $N_H(t) = \#$  of events occurred at or before time  $t$ .  $\{N_H(t), t \geq 0\}$  is a renewal processes (why?).
- ▶ What is the mean length of the interarrival times for  $\{N_H(t), t \geq 0\}$ ?

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## Revision of Example 7.7 (Cont'd)

Let  $T_k =$  the # of tosses required to get  $k$  heads in a row. To get  $k$  consecutive heads, one must first get  $k - 1$  consecutive heads, which takes  $T_{k-1}$  steps. So

$$T_k = \begin{cases} T_{k-1} + 1 & \text{w/ prob. } p, \quad (\text{if heads in the next toss}) \\ T_{k-1} + 1 + T'_k & \text{w/ prob. } 1 - p \quad (\text{if tails in the next toss}). \end{cases}$$

Here  $T'_k \sim T_k$  and  $T'_k$  is independent of the past, and hence is independent of  $T_{k-1}$ . So

$$\begin{aligned} \mathbb{E}[T_k | T_{k-1}] &= T_{k-1} + 1 + (1 - p)\mathbb{E}[T'_k | T_{k-1}] \\ &= T_{k-1} + 1 + (1 - p)\mathbb{E}[T'_k] \quad (\text{since } T'_k, T_{k-1} \text{ are indep.}) \\ &= T_{k-1} + 1 + (1 - p)\mathbb{E}[T_k] \quad (\text{since } T'_k \sim T_k) \end{aligned}$$

and hence

$$\mathbb{E}[T_k] = \mathbb{E}[\mathbb{E}[T_k | T_{k-1}]] = \mathbb{E}[T_{k-1}] + 1 + (1 - p)\mathbb{E}[T_k]$$

Subtracting  $(1 - p)\mathbb{E}[T_k]$  from both sides we get

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1, \quad k = 2, 3, 4, \dots$$

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## Revision of Example 7.7 (Cont'd)

Observe that  $T_1$  has a geometric distribution and hence  $\mathbb{E}[T_1] = 1/p$ . Using the iterative relation, we get that

$$\mathbb{E}[T_2] = (\mathbb{E}[T_1] + 1)/p = 1/p^2 + 1/p$$

$$\mathbb{E}[T_3] = (\mathbb{E}[T_2] + 1)/p = 1/p^3 + 1/p^2 + 1/p$$

⋮

$$\mathbb{E}[T_k] = (\mathbb{E}[T_{k-1}] + 1)/p = 1/p^k + \dots + 1/p^2 + 1/p = \frac{1 - p^k}{p^k(1 - p)}$$

By Proposition 7.1, we know

$$\lim_{t \rightarrow \infty} \frac{N_H(t)}{t} = \frac{1}{\mathbb{E}[T_k]} = \frac{p^k(1 - p)}{1 - p^k} = \frac{p^k q}{1 - p^k}.$$

Similarly, considering the renewal process

$$N_T(t) = \# \text{ of times to see } k \text{ tails in a row by time } t.$$

By Proposition 7.1, we also have  $\lim_{t \rightarrow \infty} \frac{N_T(t)}{t} = \frac{q^k p}{1 - q^k}$ .

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## Example 7.7 Solution (Cont'd)

Consider the counting process  $N(t) = N_H(t) + N_T(t)$ .

- ▶  $\{N(t) : t \geq 0\}$  is also a renewal process (Why?)  
Here an event occurs whenever one gets  $k$  heads in a row or  $k$  tails in a row.
- ▶ As  $t \rightarrow \infty$ ,

$$\frac{N(t)}{t} = \frac{N_H(t)}{t} + \frac{N_T(t)}{t} \rightarrow \frac{p^k q}{1 - p^k} + \frac{q^k p}{1 - q^k}.$$

which is the reciprocal of mean length  $\mathbb{E}[T]$  of interarrival times for  $\{N(t) : t \geq 0\}$ .

- ▶ From the above we can answer the 2nd question: the expected number of tosses required to get  $k$  heads in a row or  $k$  tails in a row is

$$\mathbb{E}[T] = \frac{1}{p^k q / (1 - p^k) + q^k p / (1 - q^k)}.$$

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## Example 7.7 Solution (Cont'd)

With probability 1, the long run proportion of events that are  $k$ -heads is

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{N_H(t)}{N(t)} &= \lim_{t \rightarrow \infty} \frac{N_H(t)}{N_H(t) + N_T(t)} = \lim_{t \rightarrow \infty} \frac{N_H(t)/t}{N_H(t)/t + N_T(t)/t} \\ &= \frac{p^k q / (1 - p^k)}{p^k q / (1 - p^k) + q^k p / (1 - q^k)} \end{aligned}$$

The probability of getting  $k$ -heads before  $k$ -tails is, by SLLN, equal to the long-run proportion of events that are  $k$ -heads.

$$P(k\text{-heads before } k\text{-tails}) = \frac{p^k q / (1 - p^k)}{p^k q / (1 - p^k) + q^k p / (1 - q^k)}$$

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## Theorem 7.2 CLT for Renewal Processes

Suppose that  $\mu$  and  $\sigma^2$  are, respectively, the mean and variance of the interarrival times of a renewal process  $\{N(t), t \geq 0\}$ . Then

$$\lim_{t \rightarrow \infty} \frac{\text{Var}(N(t))}{t} = \frac{\sigma^2}{\mu^3},$$

and  $N(t)$  is asymptotically  $N(t/\mu, \sigma^2 t/\mu^3)$ , i.e.,

$$\lim_{t \rightarrow \infty} P\left(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) = \Phi(x)$$

where  $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{\pi}} e^{-z^2/2} dz$  is the CDF of the standard normal distribution.

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## Proof of Theorem 7.2

$$\begin{aligned} & P\left(\frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x\right) \\ &= P\left(N(t) < \frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x\right) \\ &= P(N(t) \leq n) && \left(\text{Let } n = \left\lfloor \frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x \right\rfloor\right) \\ &= P(S_n \geq t) && (\text{Recall } N(t) \leq n \Leftrightarrow S_n \geq t) \\ &= P\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \geq \frac{t - n\mu}{\sqrt{n}\sigma}\right) \\ &\rightarrow 1 - \Phi\left(\frac{t - n\mu}{\sqrt{n}\sigma}\right) && \text{as } n \rightarrow \infty \text{ by the CLT for } S_n \\ &= \Phi\left(-\frac{t - n\mu}{\sqrt{n}\sigma}\right) && (\text{since } 1 - \Phi(z) = \Phi(-z)) \end{aligned}$$

Here  $\lfloor y \rfloor$  means the greatest integer less or equal to  $y$   
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## Proof of Theorem 7.2 (Cont'd)

It remains to show that

$$-\frac{t - n\mu}{\sqrt{n}\sigma} \rightarrow x \quad \text{as } t \rightarrow \infty.$$

Since  $n \leq \frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x$ , as  $t \rightarrow \infty$ , we have

$$\frac{t - n\mu}{\sqrt{n}\sigma} \geq \frac{t - \left(\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x\right)\mu}{\sigma \sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x}} = \frac{-\sigma x \sqrt{t/\mu}}{\sigma \sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x}} \rightarrow -x.$$

Similarly because  $n \geq \frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} - 1$ , we have

$$\frac{t - n\mu}{\sqrt{n}\sigma} \leq \frac{t - \left(\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x - 1\right)\mu}{\sigma \sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x}} = \frac{-\sigma x \sqrt{t/\mu} - \mu}{\sigma \sqrt{\frac{t}{\mu} + \frac{\sigma}{\mu} \sqrt{\frac{t}{\mu}} x} - 1} \rightarrow -x$$

as  $t \rightarrow \infty$ .

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