6.5. Limiting Probabilities

Definition. Just like discrete-time Markov chains, if the probability that a continuous-time Markov chain will be in state \( j \) at time \( t \), \( P_j(t) \), converges to a limiting value \( P_j \) independent of the initial state \( i \), for all \( i \in \mathcal{X} \)

\[
P_j = \lim_{t \to \infty} P_j(t) > 0
\]
then we say \( P_j \) is the limiting probability of state \( j \). If \( P_j \) exists for all \( j \in \mathcal{X} \), we say \( \{P_j\}_{j \in \mathcal{X}} \) is the limiting distribution of the chain.

Remark. If \( \lim_{t \to \infty} P_j(t) \) exists, we must have

\[
\lim_{t \to \infty} P_j'(t) = 0.
\]

Interpretation of the Balanced Equations

\[
\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj}
\]

\[
\nu_j P_j = \text{rate at which the process leaves state } j
\]

\[
\sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} = \text{rate at which the process enters state } j
\]

Balanced equations means that the rates at which the process enters and leaves state \( j \) are equal.

The limiting distribution \( \{P_j\}_{j \in \mathcal{X}} \) can be obtained by solving the balanced equations along with the equation \( \sum_{j \in \mathcal{X}} P_j = 1 \).

Remarks. Just like discrete-time Markov chains, a sufficient condition for the existence of a limiting distribution is that the chain is irreducible and positive recurrent.

Examples

- Poisson processes: \( \mu_n = 0, \lambda_n = \lambda \) for all \( n \geq 0 \)
  \[
  \nu_i = \lambda, \quad \nu_{i+1} = 1, \quad q_{i+1,i} = \nu_{i+1} = \lambda
  \]
  Balanced equations:
  \[
  \nu_j P_j = P_{j-1} q_{j-1,j} \Rightarrow \lambda P_j = \lambda P_{j-1} \Rightarrow P_j = P_j
  \]
  No limiting distribution exists. The chain is not irreducible. All states are transient.

- Pure birth processes with \( \lambda_n > 0 \) for all \( n \)
  No limiting distribution exists. The chain is not irreducible. All states are transient.

- Pure birth processes with \( \lambda_n > 0 \) for \( n \leq 10 \), and \( \lambda_n = 0 \) for all \( n > 10 \).
  State space \( \mathcal{X} = \{0, 1, 2, \ldots, 10\} \).
  State 10 is the only absorbing state. All others are transient.

Variation of Example 5.5 (M/M/1 Queueing)

- single-server service station
- Poisson arrival of customers with rate \( \lambda \)
- Upon arrival, customer
  - goes into service if the server is free (queue length = 0)
  - joins the queue if \( 1 \leq \text{queue length} < N \), or
  - walks away if queue length \( \geq N \)
- Service times are i.i.d. \( \sim \text{Exp}(\mu) \)

Q: What fraction of potential customers are lost?

\( X(t) \) = \# of customers in the station. This is a birth-death process with

\[
\begin{align*}
\mu_0 &= 0, \\
\mu_n &= \mu, 1 \leq n \leq N, \\
\lambda_n &= \lambda, 0 \leq n < N, \\
\lambda_N &= 0
\end{align*}
\]

Recall the forward equations are

\[
P_j'(t) = \left( \sum_{k \in \mathcal{X}, k \neq j} P_k(t) q_{kj} \right) - \nu_j P_j(t)
\]
If we let \( t \to \infty \), and assume that we can interchange limit and summation, we obtain

\[
\lim_{t \to \infty} P_j'(t) = \lim_{t \to \infty} \left( \sum_{k \in \mathcal{X}, k \neq j} P_k(t) q_{kj} \right) - \nu_j P_j(t)
\]

\[
= \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} - \nu_j P_j
\]

Hence we get the balanced equations.

\[
\nu_j P_j = \sum_{k \in \mathcal{X}, k \neq j} P_k q_{kj} \quad \text{for all } j \in \mathcal{X}
\]

Examples

- Poisson processes: \( \mu_n = 0, \lambda_n = \lambda \) for all \( n \geq 0 \)
  \[
  \nu_i = \lambda, \quad \nu_{i+1} = 1, \quad q_{i+1,i} = \nu_{i+1} = \lambda
  \]
  Balanced equations:
  \[
  \nu_j P_j = P_{j-1} q_{j-1,j} \Rightarrow \lambda P_j = \lambda P_{j-1} \Rightarrow P_j = P_j
  \]
  No limiting distribution exists. The chain is not irreducible. All states are transient.

- Pure birth processes with \( \lambda_n > 0 \) for all \( n \)
  No limiting distribution exists. The chain is not irreducible. All states are transient.

- Pure birth processes with \( \lambda_n > 0 \) for \( n \leq 10 \), and \( \lambda_n = 0 \) for all \( n > 10 \).
  State space \( \mathcal{X} = \{0, 1, 2, \ldots, 10\} \).
  State 10 is the only absorbing state. All others are transient.

Variation of Example 5.5 (M/M/1 Queueing)

Balanced equations

\[
\begin{align*}
\lambda P_0 &= \mu P_0 \\
(\mu + \lambda) P_n &= \lambda P_{n-1} + \mu P_{n+1}, \quad 1 \leq n \leq N - 1 \\
\mu P_N &= \lambda P_{N-1} \\
P_1 &= (\lambda / \mu) P_0 \\
P_2 &= (\lambda / \mu) P_1 = (\lambda / \mu)^2 P_0 \\
\vdots
\end{align*}
\]

Plugging \( P_j = (\lambda / \mu)^j P_0 \) to \( \sum_{j=1}^{N} P_j = 1 \), one can solve for \( P_0 \) and get

\[
P_i = \frac{1 - \lambda / \mu}{1 - (\lambda / \mu)^{N-i}} (\lambda / \mu)^i
\]

Ans: Fraction of customers lost = \( P_N = \frac{1 - \lambda / \mu}{1 - (\lambda / \mu)^{N}} (\lambda / \mu)^N \)
Limiting Distribution for Birth and Death Processes

For a birth and death process,

\[ \nu_0 = \lambda_0, \]
\[ \nu_i = \lambda_i + \mu_i, \ i > 0 \]
\[ p_{01} = 1, \]
\[ p_{i+1} = \frac{\lambda_i}{\mu_i} p_i, \ i > 0 \]
\[ q_{i-1} = \frac{\mu_i}{\mu_i} p_{i+1}, \ i > 0 \]
\[ p_{i,j} = 0 \] if \(|i - j| > 1\)

Balanced equations

\[ \lambda_0 p_0 = \mu_1 p_1 \]
\[ (\mu_i + \lambda_i)p_i = \lambda_{i-1}p_{i-1} + \mu_{i+1}p_{i+1}, \ i > 0 \]

\[ \Rightarrow \lambda_n p_n = \mu_{n+1} p_{n+1}. \]
\[ p_n = \frac{\lambda_{n-1} p_{n-1}}{\mu_n} = \frac{\lambda_{n-1} \lambda_{n-2} \cdots \lambda_0}{\mu_1 \mu_2 \cdots \mu_n} p_0 \]

Lecture 13 - 7

Lemma: (Ratio Test) If \( a_n \geq 0 \) for all \( n \), then

\[ \sum_{n=1}^{\infty} a_n < \infty \] if \( \lim_{n \to \infty} a_n/a_{n-1} < 1 \)

For \( a_n = \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \), \( a_n/a_{n-1} = \frac{\lambda_{n-1}}{\mu_n} \). By the ratio test, if

\[ \lim_{n \to \infty} \frac{\lambda_{n-1}}{\mu_n} < 1, \]

then \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} < \infty \), the limiting distribution exists.

Example 6.4 Linear Growth Model with Immigration

\[ \mu_n = n\mu, \ \lambda_n = n\lambda + \theta \]

Using the Ratio Test,

\[ \lim_{n \to \infty} \frac{\lambda_{n-1}}{\mu_n} = \lim_{n \to \infty} \frac{(n-1)\lambda + \theta}{n\mu} = \frac{\lambda}{\mu} \]

So the linear growth model has a limiting distribution if and only \( \lambda < \mu \).

Lecture 13 - 9

Limiting Dist’n for Birth and Death Processes (Cont’d)

Using the fact \( \sum_{n=0}^{\infty} p_n = 1 \), we obtain

\[ P_0 + \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_2 \lambda_1}{\mu_1 \mu_2 \cdots \mu_n} p_0 = 1 \]

One can see that, to have a limiting distribution, it is necessary that

\[ \sum_{n=1}^{\infty} \frac{\lambda_{n-1} \cdots \lambda_2 \lambda_1}{\mu_1 \mu_2 \cdots \mu_n} < \infty \]

This condition also may be shown to be sufficient.

If that is finite we can see that the limiting distribution is

\[ P_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \frac{\lambda_0 \lambda_1 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n}} \]

Lecture 13 - 8

Duration Times for Birth and Death Processes

Let

\[ T_i = \text{time to move from state } i \text{ to state } i+1, \ i = 0, 1, \ldots \]

Then \( T_i, i = 0, 1, \ldots \) are independent random variables.

\[ \mathbb{E}[T_i] = \mathbb{E}[T_i | i \to i+1] p_{i,i+1} + \mathbb{E}[T_i | i \to i-1] p_{i,i-1} \]

\[ = \frac{1}{\lambda_i + \mu_i} + \left( \frac{1}{\lambda_i + \mu_i} + \mathbb{E}[T_{i-1}] + \mathbb{E}[T_i] \right) \frac{\mu_i}{\lambda_i + \mu_i} \]

We obtain the recursive formula

\[ \lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}] \]

Lecture 13 - 10

Duration Times for Birth and Death Processes (Cont’d)

Since \( T_0 \sim \text{Exp}(\lambda_0) \), \( \mathbb{E}[T_0] = 1/\lambda_0 \).

Using the recursive formula \( \lambda_i \mathbb{E}[T_i] = 1 + \mu_i \mathbb{E}[T_{i-1}] \), we have

\[ \mathbb{E}[T_0] = \frac{1}{\lambda_0} \]
\[ \mathbb{E}[T_1] = \frac{1 + \mu_1 \mathbb{E}[T_0]}{\lambda_1 + \mu_1} = \frac{1 + \frac{\mu_1}{\lambda_1}}{\lambda_1} \frac{1}{\lambda_1} \]
\[ \mathbb{E}[T_2] = \frac{1 + \mu_2 (\frac{1}{\lambda_1} + \frac{\mu_1}{\lambda_1 \lambda_0})}{\lambda_2 + \mu_2 \frac{1}{\lambda_1} + \mu_2 \frac{\mu_1}{\lambda_1 \lambda_0}} \]

\[ \vdots \]
\[ \mathbb{E}[T_i] = \frac{1 + \mu_i \mathbb{E}[T_{i-1}]}{\lambda_i + \mu_i} = \frac{1 + \frac{\mu_i}{\lambda_i} \mathbb{E}[T_{i-1}]}{\lambda_i} \frac{1}{\lambda_i} \]

\[ = \frac{1}{\lambda_i} \left( 1 + \sum_{k=1}^{i} \frac{\mu_i \cdots \mu_{i-k+1}}{\lambda_i \cdots \lambda_{i-k}} \right) \]

Lecture 13 - 11

6.6. Time Reversibility

Definition. A continuous-time Markov chain with state space \( \chi \) is time reversible if

\[ p_{ij} = p_{ji}, \ \text{for all } i, j \in \chi \] (detailed balanced equation)

If a distribution \( \{ P_j \} \) on \( \chi \) satisfies the detailed balanced equation, then it is a stationary distribution for the process.

Example. One can use the detailed balanced equation to find the stationary distribution of a Birth and Death process.

\[ \lambda_n p_n = \mu_{n+1} p_{n+1}, \ n \geq 0 \]