Chapter 6 Continuous-Time Markov Chains

Lecture 12 - 1

6.2 Continuous-Time Markov Chains (CTMC)

**Definitions.** A stochastic process \( \{X(t), t \geq 0\} \) with state space \( \mathcal{X} \) is called a **continuous-time Markov chain** if for any two states \( i, j \in \mathcal{X} \),

\[
P(X(t+s) = j | X(s) = i, X(u) = x(u), \text{for } 0 \leq u < s) = P(X(t+s) = j | X(s) = i)
\]

for future \( s \), present \( u \), and past \( x(u) \).

If \( P(X(t+s) = j | X(s) = i) \) does not depend on \( s \) for all \( i, j \in \mathcal{X} \), then it is denoted as

\[
P_j(t) = P(X(t+s) = j | X(s) = i),
\]

and we say the CTMC is **homogeneous** in time.

In STAT253/317, we focus on homogeneous CTMC only.

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6.3 Birth and Death Processes

Let \( \{X(t), t \geq 0\} \) be a homogeneous continuous-time Markov chain. For \( i \in \mathcal{X} \), let \( T_i \) denote the amount of time that \( X(t) \) stays in state \( i \) before making a transition into a different state.

**Claim:** \( T_i \) has the **memoryless property**.

\[
P(T_i \geq t+s | T_i \geq s) = P(X(u) = i, \text{for } s \leq u \leq s+t | X(u) = i, \text{for } 0 \leq u \leq s)
\]

\[
= P(X(u) = i, \text{for } s \leq u \leq s+t | X(s) = i) \quad (\text{Markov property})
\]

\[
= P(X(u) = i, \text{for } 0 \leq u \leq t | X(0) = i) \quad (\text{time homogeneity})
\]

\[
= P(T_i \geq t) \quad \Rightarrow \quad \text{So } T_i \text{ is memoryless.}
\]

Recall that the exponential distribution is the only continuous distribution having the memoryless property. Thus \( T_i \sim \text{Exp}(\nu_i) \) for some rate \( \nu_i \).

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Examples of Birth and Death Processes

- **Poisson Processes:** \( \nu_n = 0, \lambda_n = \lambda \) for all \( n \geq 0 \)
- **Pure Birth Process:**
  \[
  \mu_n = 0 \quad \Rightarrow \quad \nu_i = \lambda_i, \quad P_{i,i+1} = 1, \quad P_{i,i-1} = 0
  \]
- **Yule Processes (Pure Birth Process with Linear Growth rate):**
  If there are \( n \) people and each independently gives birth at an exponential rate \( \lambda \), then the total rate at which births occur is \( n\lambda \).

\[
\mu_n = 0, \quad \lambda_n = n\lambda
\]
- **Linear Growth Model with Immigration:**
  \[
  \nu_n = n\mu, \quad \lambda_n = n\lambda + \theta
  \]
- **M/M/s Queueing Model:**
  \[
  s \text{ servers}
  \]
  - Poisson arrival of customers, rate = \( \lambda \)
  - Exponential service time, rate = \( \mu \)
  \[
  \Rightarrow \text{a birth and death process with constant birth rate } \lambda_n = \lambda, \text{ and death rate } \mu_n = \min(n,s)\mu.
  \]
6.4 The Transition Probability Function $P_{ij}(t)$

Recall the transition probability function $P_{ij}(t)$ of a CTMC $\{X(t), t \geq 0\}$ is

$$P_{ij}(t) = P(X(t + s) = j | X(s) = i)$$

**Example.** (Poisson Processes with rate $\lambda$)

$$P_{ij}(t) = P(N(t + s) = j | N(s) = i) = e^{-\lambda t} \frac{\lambda^j}{j!}$$

**Properties of Transition Probability Functions**

- $P_{ij}(t) \geq 0$ for all $i, j \in \mathcal{X}$ and $t \geq 0$
- (Row sums are 1) $\sum_j P_{ij}(t) = 1$ for all $i \in \mathcal{X}$ and $t \geq 0$

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**Lemma 6.2**

For any $i, j \in \mathcal{X}$, we have

(a) $\lim_{h \to 0} \frac{1 - P_{ij}(h)}{h} = \nu_i$  
(b) $\lim_{h \to 0} \frac{P_{ij}(h)}{h} = \nu_i P_{ij}$ defined as $q_{ij}$, 

where $q_{ij} = \nu_i P_{ij}$ is called the instantaneous transition rates.

**Proof.** (a) Let $T_i$ be the amount of time the process stays in state $i$ before moving to other states.

$$P_{ii}(h) = P(X(h) = i | X(0) = i)$$

$$= P(X(h) = i, \text{no transition in (0,h]} | X(0) = i)$$

$$+ P(X(h) = i, 2 \text{or more transition in (0,h]} | X(0) = i)$$

$$= P(T_i > h) + o(h) = e^{-\nu_i h} + o(h) = 1 - \nu_i h + o(h)$$

(b) $P_{ij}(h) = P(X(h) = j | X(0) = i)$

$$= P(X(h) = j, 1 \text{transition in (0,h]} | X(0) = i)$$

$$+ P(X(h) = j, 2 \text{or more transition in (0,h]} | X(0) = i)$$

$$= P(T_i < h) P_{ij} + o(h) = (1 - e^{-\nu_i h}) P_{ij} + o(h) = \nu_i P_{ij} h + o(h)$$

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**Theorem 6.1 Kolmogorov’s Backward Equations**

From Lemma 6.3 (Chapman-Kolmogorov equations), we obtain

$$P_{ij}(h + t) - P_{ij}(t) = \sum_{k \in \mathcal{X}} P_{ik}(h) P_{kj}(t) - P_{ij}(t)$$

and thus

$$\lim_{h \to 0} \frac{P_{ij}(h + t) - P_{ij}(t)}{h} = \sum_{k \in \mathcal{X}, k \neq i} P_{ik}(h) P_{kj}(t) - \frac{1 - P_{ii}(h)}{h} P_{ij}(t)$$

Now assuming that we can interchange the limit and the summation in the preceding and applying Lemma 6.2, we obtain

$$P_{ij}(t) = \sum_{k \in \mathcal{X}, k \neq i} q_{ik} P_{kj}(t) - \nu_i P_{ij}(t)$$

It turns out that this interchange can indeed be justified.

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**Theorem 6.2 Kolmogorov’s Forward Equations**

From Lemma 6.3 (Chapman-Kolmogorov equations), we obtain

$$P_{ij}(t + h) - P_{ij}(t) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(h) - P_{ij}(t)$$

and thus

$$\lim_{h \to 0} \frac{P_{ij}(t + h) - P_{ij}(t)}{h} = \lim_{h \to 0} \left\{ \sum_{k \in \mathcal{X}, k \neq j} P_{ik}(t) \frac{P_{kj}(h)}{h} - \frac{1 - P_{ij}(h)}{h} P_{ij}(t) \right\}$$

Now assuming that we can interchange the limit and the summation in the preceding and applying Lemma 6.2, we obtain

$$P_{ij}'(t) = \sum_{k \neq j} P_{ik}(t) q_{kj} - \nu_j P_{ij}(t)$$

Unfortunately, this interchange is not always justifiable. However, the forward equations do hold in most models, including all birth and death processes and all finite state models.

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**Lecture 12 - 7**

**Lemma 6.3 Chapman-Kolmogorov Equation**

For all $i, j \in \mathcal{X}$ and $t \geq 0$,

$$P_{ij}(t + s) = \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s)$$

**Proof.**

$$P_{ij}(t + s) = P(X(t + s) = j | X(0) = i)$$

$$= \sum_{k \in \mathcal{X}} P(X(t + s) = j, X(t) = k | X(0) = i)$$

$$= \sum_{k \in \mathcal{X}} P(X(t) = k | X(0) = i) P(X(t) = j | X(t) = k)$$

$$= \sum_{k \in \mathcal{X}} P(X(t) = k) P(X(t) = j | X(t) = k)$$

(Markov Property)

$$= \sum_{k \in \mathcal{X}} P_{ik}(t) P_{kj}(s)$$

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**Recall that we define the instantaneous transition rates**

$$q_{ij} = \nu_i P_{ij} \quad \text{for } i, j \in \mathcal{X}, i \neq j$$

If we define $q_{ii}$ as $-\nu_i$. For finite state space case $\mathcal{X} = \{1,2,\ldots,m\}$, define the matrices

$$P(t) = \begin{bmatrix} P_{11}(t) & \cdots & P_{1m}(t) \\ \vdots & \ddots & \vdots \\ P_{m1}(t) & \cdots & P_{mm}(t) \end{bmatrix}, \quad P'(t) = \begin{bmatrix} P'_{11}(t) & \cdots & P'_{1m}(t) \\ \vdots & \ddots & \vdots \\ P'_{m1}(t) & \cdots & P'_{mm}(t) \end{bmatrix}$$

$$Q = \begin{bmatrix} q_{11} & \cdots & q_{1m} \\ \vdots & \ddots & \vdots \\ q_{m1} & \cdots & q_{mm} \end{bmatrix} = \begin{bmatrix} -\nu_1 & \nu_1 \nu_2 & \cdots & \nu_1 \nu_m \\ \nu_2 \nu_1 & -\nu_2 & \cdots & \nu_2 \nu_m \\ \vdots & \ddots & \ddots & \vdots \\ \nu_m \nu_{m-1} & \cdots & -\nu_m \end{bmatrix}$$

In matrix notation,

Forward equation: $P'(t) = P(t)Q$

Backward equation: $P'(t) = Q^{-1}P(t)$

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**Lecture 12 - 12**