

Stat 317/253 Winter 2014 HW #4 January 17

Due Friday **January 24th, in class** (at the beginning of the lecture period)

Readings: [IPM10e] Section 4.4, p.214-230 (skip Example 4.26 on p.225-228)

Problems to Turn In:

1. [IPM10e] Exercise 4.15
2. (This problem is to show that **an irreducible finite-state Markov chain is positive recurrent.**) Let $\{X_n, n \geq 0\}$ be an irreducible Markov chain with M states ($M < \infty$).
 - (a) Use Problem 1 ([IPM10e] Exercise 4.15) above to show that there exists a $\delta > 0$ such that for all i , the probability of reaching j some time in the first M steps, starting at i , is greater than δ .
 - (b) Use part (a) to show that there exist $C < \infty$ and $\rho < 1$ such that for any states i, j ,
$$P(X_m \neq j, m = 0, \dots, n | X_0 = i) < C\rho^n.$$
 - (c) Use part (b) to show that $\mathbb{E}(T_{ij}) < \infty$, where T_{ij} is the first time that the Markov chain reaches the state j when starting from state i . This implies that state j is positive recurrent.
3. [IPM10e] Exercise 4.38
4. [IPM10e] Exercise 4.39

Problems for Self-Study (Do NOT turn in):

1. [IPM10e] Exercise 4.16 (See p.748 solutions for solutions)
2. (**Must See!**) [IPM10e] Exercise 4.10, 4.24, 4.46
The exercises are about on finding limit distributions for finite-state Markov chains. Problems of this type always appear in the **midterm**. Make sure you are familiar with them. Solutions are included for self-study. Questions? Ask TA or Yibi.

[IPM10e] Exercise 4.20

A transition probability matrix \mathbf{P} is said to be doubly stochastic if the sum over each column equals one; that is,

$$\sum_i P_{ij} = 1, \quad \text{for all } j$$

If such a chain is irreducible and aperiodic and consists of $M + 1$ states $0, 1, \dots, M$, show that the limiting probabilities are given by

$$\pi_j = \frac{1}{M + 1}, \quad j = 0, 1, \dots, M$$

Solution: We can see that $\pi_j = \frac{1}{M+1}, j = 0, 1, \dots, M$ is a stationary distribution since

$$\begin{aligned}\sum_{j=0}^M \pi_i P_{ij} &= \sum_{j=0}^M \frac{1}{M+1} P_{ij} \\ &= \frac{1}{M+1} \sum_{j=0}^M P_{ij} = \frac{1}{M+1} \times 1 \quad (\text{since } \mathbf{P} \text{ is doubly stochastic}) \\ &= \frac{1}{M+1} = \pi_j\end{aligned}$$

Such a Markov chain is positive recurrent since it is finite and irreducible. Along with aperiodicity, we know that the chain is ergodic. By Theorem 4.1, such a chain will have a limiting distribution, and the stationary distribution $\pi_j = \frac{1}{M+1}, j = 0, 1, \dots, M$ is the unique limiting distribution

[IPM10e] Exercise 4.24

Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

Solution: Let X_n be the color of the ball in the n th draw. The state space is $\{R, W, B\}$. The transition probability matrix is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} R & W & B \end{matrix} \\ \begin{matrix} R \\ W \\ B \end{matrix} & \begin{pmatrix} \frac{1}{5} & 0 & \frac{4}{5} \\ \frac{3}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{3}{9} & \frac{4}{9} & \frac{2}{9} \end{pmatrix} \end{matrix}$$

We can get the long-run proportion of the 3 colors drawn by solving the system of equations $\pi\mathbf{P} = \pi$, where $\pi = (\pi_R, \pi_W, \pi_B)$

$$\begin{aligned}\pi_R &= \frac{1}{5}\pi_R + \frac{2}{7}\pi_W + \frac{3}{9}\pi_B \\ \pi_W &= \frac{3}{7}\pi_W + \frac{4}{9}\pi_B \\ \pi_B &= \frac{4}{5}\pi_R + \frac{2}{7}\pi_W + \frac{2}{9}\pi_B\end{aligned}$$

and also that $\pi_R + \pi_W + \pi_B = 1$. The solution is

$$\pi_R = \frac{25}{89}, \quad \pi_W = \frac{28}{89}, \quad \pi_B = \frac{36}{89}$$

In the long run, $25/89 \approx 28.1\%$ of the balls drawn will be red, $28/89 \approx 31.5\%$ be white, $36/89 \approx 40.4\%$ be blue.

[IPM10e] Exercise 4.46

An individual possesses r umbrellas which he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability p .

- (i) Define a Markov chain with $r + 1$ states which will help us to determine the proportion of time that our man gets wet. (Note: He gets wet if it is raining, and all umbrellas are at his other location.)
- (ii) Show that the limiting probabilities are given by

$$\pi_i = \begin{cases} \frac{q}{r+q}, & \text{if } i = 0 \\ \frac{1}{r+q}, & \text{if } i = 1, \dots, r \end{cases} \quad \text{where } q = 1 - p.$$

- (iii) What fraction of time does our man get wet?
- (iv) When $r = 3$, what value of p maximizes the fraction of time he gets wet?

Solution:

- (i) Let X_n be the number of umbrellas in his current location. The state space is $\mathcal{X} = \{0, 1, \dots, r\}$.

If the man has $i > 0$ umbrellas in his current location, then there will be $r - i$ umbrellas in the other place, when he gets to the other place, there will still be $r - i$ umbrellas there if it doesn't rain. So $P_{i,r-i} = q$. If it rains, then he will bring one umbrella there, so there will be $r - i + 1$ umbrellas, $P_{i,r-i+1} = p$. If he has no umbrella at hand, then there will be r umbrellas in the other place whether it rains or not. So $P_{0r} = 1$. As a summary, the transition probabilities are

$$\begin{aligned} P_{i,r-i} = q \quad P_{i,r-i+1} = p, & \quad \text{for } i = 1, 2, \dots, r \\ P_{0r} = 1 \\ P_{ij} = 0 & \quad \text{if } i + j > r + 2 \text{ or } i + j < r \end{aligned}$$

- (ii) This Markov chain is irreducible and finite, and hence positive recurrent. Along with aperiodicity, we know that the only stationary distribution is the limiting distribution. So we just need to show that the π_i 's given in the problem is a stationary distribution.

$$\begin{aligned} \sum_{i=0}^m \pi_i P_{i0} &= \pi_r P_{r0} = \frac{1}{r+q} \times q = \frac{q}{r+q} = \pi_0 \\ \sum_{i=0}^m \pi_i P_{ij} &= \pi_{r-j} P_{r-j,j} + \pi_{r-j+1} P_{r-j+1,j} \\ &= \frac{1}{r+q} \times q + \frac{1}{r+q} \times p = \frac{1}{r+q} = \pi_j \quad \text{for } 1 \leq j \leq r-1 \\ \sum_{i=0}^m \pi_i P_{ir} &= \pi_0 P_{0r} + \pi_1 P_{1r} = \frac{q}{r+q} \times 1 + \frac{1}{r+q} \times p = \frac{1}{r+q} = \pi_r \end{aligned}$$

Thus the π_i 's given is indeed the stationary distribution, and hence is the limiting distribution.

(iii) The man gets wet only when he has no umbrella at his current location, and it rains. So the probability is $\pi_0 p = \frac{pq}{r+q}$.

(iv) Let $f(p) = \frac{pq}{r+q} = \frac{p(1-p)}{r+1-p}$. So

$$f'(p) = \frac{(1-2p)(r+1-p) + p(1-p)}{(r+1-p)^2} = \frac{(r+1-p)^2 - r(r+1)}{(r+1-p)^2} = 1 - \frac{r(r+1)}{(r+1-p)^2}.$$

The two roots of $f'(p) = 0$ are $r+1 \pm \sqrt{r(r+1)}$. Since $0 \leq p \leq 1$, $p = r+1 - \sqrt{r(r+1)}$ is what we want. Since

$$f''(p) = \frac{-2r(r+1)}{(r+1-p)^3} < 0,$$

$p = r+1 - \sqrt{r(r+1)}$ must be a maximum. When $r = 3$, the maximum is attained at $4 - 2\sqrt{3}$.
