Statistics 253/317
Introduction to Probability Models

Winter 2014 - Midterm Exam
Monday, Feb 10, 2014

Student Name (print): ________________________________

(a) Do not sit directly next to another student.

(b) This is a closed-book, closed-note examination. You should have your hand calculator and one letter-sized of formula sheet that you may refer to.

(c) *You need to show your work to receive full credit. In particular, if you are basing your calculations on a formula or an expression (e.g.,* $\mathbb{E}(Y|X = k)$*), write down that formula before you substitute numbers into it.*

(d) If a later part of a question depends on an earlier part, the later part will be graded conditionally on how you answered the earlier part, so that a mistake on the earlier part will not cost you all points on the later part. *If you can't work out the actual answer to an earlier part, put down your best guess and proceed.*

(e) Do not pull the pages apart. If a page falls off, sign the page. If you do not have enough room for your work in the place provided, ask for extra papers, label and sign the pages.

Question	*Points Available*	*Points Earned*
1	25	
2	30	
3	15	
4	30	
TOTAL	100	

Problem 1. **[25 points]** A Markov chain $\{X_n : n \geq 0\}$ on the state space $\{0, 1, 2, 3, 4, 5\}$ has transition matrix

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \left(\begin{array}{cccccc} 0.5 & 0.5 & 0 & 0 & 0 & 0 \\ 0.3 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.9 & 0 \\ 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 \\ 0 & 0 & 0.6 & 0 & 0.3 & 0.1 \\ 0 & 0.2 & 0 & 0.2 & 0.2 & 0.4 \end{array} \right) \end{array}$$

(a) [10 pts] Find all communicating classes. For each state, determine whether it is recurrent or transient. Explain your reasoning.

Answer:

$$\begin{array}{ccccc} 0 & \leftarrow & 3 & & 2 \\ \updownarrow & \swarrow & \updownarrow & \searrow & \updownarrow \\ 1 & \leftarrow & 5 & \leftrightarrow & 4 \end{array}$$

Clearly, $\{0, 1\}$ is a closed class because and $0 \leftrightarrow 1$ and the chain will never leave the class once entered. $\{2, 3, 4, 5\}$ is a class since $3 \leftrightarrow 5 \leftrightarrow 4 \leftrightarrow 2$.

The class $\{0, 1\}$ is recurrent since the chain will never leave the class. The class $\{2, 3, 4, 5\}$ is transient since there is a positive chance to leave the class, and once left the chain will never go back to $\{2, 3, 4, 5\}$.

(b) [5 pts] Suppose the Markov chain starts from state 5. What is the long run proportion of time that the Markov chain is state 5?

Answer: Since state 5 is transient, it will only be visited finite number of times. So the long run proportion of time that state 5 is visited is 0.

(c) [10 pts] Find $\mathrm{P}(X_1 \geq 4, X_2 \geq 4, X_3 \geq 4, X_4 = 4 | X_0 = 5)$.

Answer: Create another process $\{W_n, n = 0, 1, 2, \ldots\}$ with an absorbing state A

$$W_n = \begin{cases} X_n & \text{if } X_k \geq 4 \text{ for all } k = 0, 1, 2, \ldots, n \\ A & \text{otherwise} \end{cases}$$

Then $\{W_n : n = 0, 1, 2, \ldots\}$ is a Markov chain with the transition matrix

$$Q = \begin{matrix} & \begin{matrix} A & 4 & 5 \end{matrix} \\ \begin{matrix} A \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0.6 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.4 \end{pmatrix} \end{matrix}$$

Observe that $\mathrm{P}(X_1 \geq 4, X_2 \geq 4, X_3 \geq 4, X_4 \geq 4 | X_0 = 5) = \mathrm{P}(W_4 = 4 | W_0 = 5) = Q^4_{54}$.

$$Q^2 = \begin{matrix} & \begin{matrix} A & 4 & 5 \end{matrix} \\ \begin{matrix} A \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0.82 & 0.11 & 0.07 \\ 0.68 & 0.14 & 0.18 \end{pmatrix} \end{matrix}$$

So

$$Q^4_{54} = (0.68, 0.14, 0.18) \begin{pmatrix} 0 \\ 0.11 \\ 0.14 \end{pmatrix} = 0.68 \times 0 + 0.14 \times 0.11 + 0.18 \times 0.14 = 0.0406.$$

Problem 2. **[30 points]** Suppose that customers arrive to a bank according to a Poisson process with rate λ. There are an infinite number of servers in this bank so a customer begins service upon arrival. The service times of the arrivals are independent exponential random variables with rate μ, and are independent of the arrival process. Customers depart the bank immediately when their service ends.

(a) [5 pts] Suppose the first customer entered the bank at time $t = 1$. What is the expected number of customers that enter the bank between time $t = 0$ till time $t = 10$, including the first one?

Answer: Let $N(t)$ be the number of customers entering the bank at or before time t, and T_1 be the arrival time of the first customer. Also that $T_1 = 1$, the first customer entered at time $t = 1$ implies $N(1) = 1$. Thus,

$$\begin{aligned}\mathbb{E}[N(10)|T_1 = 1] &= \mathbb{E}[N(1)|T_1 = 1] + \mathbb{E}[N(10) - N(1)|T_1 = 1] \\ &= 1 + \mathbb{E}[N(10) - N(1)|T_1 = 1]\end{aligned}$$

By the independent increment property of the Poisson process, $N(10) - N(1)$ is independent of what happened in the time interval $[0, 1]$. So,

$$\mathbb{E}[N(10) - N(1)|T_1 = 1] = \mathbb{E}[N(10) - N(1)] = 9\lambda,$$

where the last identity comes from the fact that

$$N(10) - N(1) \sim Poisson(\lambda(10 - 1)) = Poisson(9\lambda).$$

So,

$$\mathbb{E}[N(10)|T_1 = 1] = 1 + 9\lambda.$$

(b) [5 pts] The first customer, of course, finds the bank empty at arrival. What is the probability that no other customers arrive when this customer is served?

Answer: Let T_2 be the interarrival time between the first and second customer. As the arrival process is Poisson with rate λ, we know

$$T_2 \sim Exp(\lambda).$$

Let S_1 be the service time of the first customer. We know the service times are exponentially distributed with rate μ

$$S_1 \sim Exp(\mu)$$

and are independent of the arrival process, so S_1 and T_2 are independent. No customer arrive during the service time of the first customer if and only if $S_1 < T_2$. So the probability is

$$\mathrm{P}(S_1 < T_2) = \frac{\mu}{\lambda + \mu}.$$

Here we use the property of exponential random variables in Section 5.2.3 of the textbook (p.302).

Define

$$N(t) = \text{the number of customers entering the bank at or before time } t,$$
$$S(t) = \text{the number of customers still being served in the bank at time } t,$$
$$D(t) = \text{the number of customers that have finished service and depart the bank at or before time } t.$$

Clearly

$$N(t) = S(t) + D(t) \quad \text{for all } t.$$

(c) [10 pts] Suppose the bank starts empty at time 0, $N(0) = S(0) = 0$. What is the probability that $S(t) = n$ for $n = 0, 1, 2, \ldots$?

Answer: For a fixed t, we categorize customers entering the bank before or at time t as Type I or Type II according to whether if they are in or not in the bank at time t. Then $S(t)$ is simply the number of Type I customers.

A customer entering the bank at time s will still be in the bank at time t if and only if

$$s + S > t$$

here S is the service time of this customer. So for a customer entering the bank at time s, the probability that he is still in the system at time t is

$$p(s) = \mathrm{P}(s + T > t) = \mathrm{P}(T > t - s) = e^{-\mu(t-s)} \qquad s < t.$$

Using the thinning property of the Poisson process, we can see that $S(t)$ has a Poisson distribution with mean

$$\Lambda \overset{\text{defined as}}{=\!=} \lambda \int_0^t p(s)ds = \lambda \int_0^t e^{-\mu(t-s)}ds = \lambda \int_0^t e^{-\mu s}ds = \frac{\lambda}{\mu}(1 - e^{-\mu t}).$$

So

$$\mathrm{P}(S(t) = n) = e^{-\Lambda}\frac{\Lambda^n}{n!} \qquad \text{for } n = 0, 1, 2, \ldots.$$

(d) [10 pts] Suppose $N(0) = 0$ and n customers have entered the bank by time t, i.e., $N(t) = n$, what is the mean of $S(t)$ for $k = 0, 1, 2, \ldots, n$?

Answer: Let $A_1 < A_2 < \ldots < A_n$ be respectively the arrival times of the first n customers. Observe that given $N(t) = n$, $S(t)$ is simply the number of customers with $A_i + S_i > t$, where S_i is the service time of the ith customer.

$$S(t) = \sum_{i=1}^{n} \mathbf{1}_{\{A_i+S_i>t\}}$$

Given $N(t) = n$ and $A_1, A_2, \ldots, A_n$, because the service time S_i's are i.i.d. Exponential with rate μ, we know

$$\begin{aligned}
\mathbb{E}[S(t)|N(t) = n, A_1, A_2, \ldots, A_n] &= \mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\{A_i+S_i>t\}}\right] \\
&= \sum_{i=1}^{n} \mathrm{P}(A_i + S_i > t) \\
&= \sum_{i=1}^{n} \mathrm{P}(S_i > t - A_i) = \sum_{i=1}^{n} e^{-\mu(t-A_i)}
\end{aligned}$$

Given $N(t) = n$, we know that $A_1, A_2, \ldots, A_n$ are distributed like the order statistic $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ from the uniform distribution on $(0, t]$. So

$$\begin{aligned}
\mathbb{E}[S(t)|N(t) = n] &= \mathbb{E}[\mathbb{E}[S(t)|N(t) = n, A_1, A_2, \ldots, A_n]|N(t) = n] \\
&= \mathbb{E}[\sum_{i=1}^{n} e^{-\mu(t-A_i)}|N(t) = n] \\
&= \mathbb{E}[\sum_{i=1}^{n} e^{-\mu(t-U_{(i)})}] \\
&= \mathbb{E}[\sum_{i=1}^{n} e^{-\mu(t-U_i)}] = \sum_{i=1}^{n} \mathbb{E}[e^{-\mu(t-U_i)}] \\
&= n\mathbb{E}[e^{-\mu(t-U_i)}] = n\int_0^t \frac{1}{t} e^{-\mu(t-u)} du \\
&= \frac{n}{\mu t}(1 - e^{-\mu t}).
\end{aligned}$$

Problem 3. **[15 points]** Consider a highly contagious disease and an infinite population of vulnerable individuals. During each time period, each infected but undiscovered individual has his/her symptoms appear with probability p and then he/she will be discovered by public health officials. Otherwise he/she stays undiscovered and infects a new person. All discovered infected individuals are quarantined so that they won't affect others anymore. Let X_n be the number of **infected** but **undiscovered** individuals in the nth period, $n = 1, 2, 3, \ldots$. Unknown to public health officials, an infected person enters the population at time period 0. Prior to that, no one is infected. So $X_0 = 1$.

(a) [5 pts] Explain why that $\{X_n, n = 0, 1, 2, \ldots\}$ is a branching process.

Answer: Observe that

$$X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i}$$

where

$$Z_{n,i} = \begin{cases} 0 & \text{with probability } p \quad \text{(if discovered)} \\ 2 & \text{with probability } 1-p \quad \text{(if not discovered)} \end{cases}$$

since in the next period each infected but undiscovered individual either ends with no offspring (if discovered) or two offsprings (him/herself + the newly infected). Also observe that $Z_{n,i}$'s are independent. So $\{X_n, n = 0, 1, 2, \ldots\}$ is a branching process.

(b) [10 pts] Observed that if $X_n = 0$ for some finite n, all infected individuals are discovered and no one else will be infected anymore, which means *the epidemic is controlled.* What is the probability that the epidemic can be controlled, i.e., the chain X_n will reach state 0 eventually?

Answer: The office spring distribution has mean

$$\mu = \mathbb{E}[Z_{n,i}] = 0 \times p + 2 \times (1-p) = 2(1-p).$$

We know that a branching process will extinct with probability 1 if $\mu \geq 1$. So for $p \geq 1/2$, the epidemic can be controlled with probability 1.

For $p < 1/2$, the extinction probability is the smallest root to the equation $g(s) = s$ in the interval (0,1), where $g(s) = \mathbb{E}[s^{Z_{n,i}}] = p + (1-p)s^2$ is the generating function of the offspring distribution. Solving the equation

$$g(s) - s = (1-p)s^2 - s + p = (s-1)[(1-p)s - p] = 0,$$

we get two roots $s = 1$ and $s = p/(1-p) < 1$. So the extinction probability in this case is $p/(1-p)$.

To sum up, the probability that the epidemic can be controlled is

$$\begin{cases} 1 & \text{if } p \geq 1/2 \\ p/(1-p) & \text{if } p < 1/2. \end{cases}$$

Problem 4. [30 points] A coin having probability p, $0 < p < 1$, of landing heads is tossed continually. We are interested in the length of consecutive heads in the tosses. Define $X_n = k$ if the coin comes up heads in all of the most recent k tosses (from the $(n-k+1)$st up to the nth), but tails in the $(n-k)$th toss. On the contrary, if the coin comes up tails in the nth toss, then let $X_n = 0$. For example, for the outcome of 15 tosses `HHHHTTHHTHTTHHH`, the value of X_n's are

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
nth toss	H	H	H	H	T	T	H	H	T	H	T	T	H	H	H
X_n	1	2	3	4	0	0	1	2	0	1	0	0	1	2	3

Observe that

$$X_n = \begin{cases} X_{n-1} + 1 & \text{with probability } p \quad \text{(if the coin comes up heads in the } n\text{th toss)} \\ 0 & \text{with probability } 1-p \quad \text{(if the coin comes up tails in the } n\text{th toss)} \end{cases}$$

So $\{X_n, n = 1, 2, 3 \ldots\}$ is a Markov chain.

(a) [4 pts] Is the Markov chain $\{X_n : n \geq 0\}$ time-reversible? Why or why not?

Answer: No, because for example $P_{k,0} > 0$ but $P_{0,k} = 0$ for $k \geq 2$.

(b) [10 pts] Show that the Markov chain $\{X_n, n = 1, 2, 3 \ldots\}$ is irreducible and find the limiting distribution

$$\pi_i = \lim_{n \to \infty} \mathrm{P}(X_n = i) \quad \text{for } i = 0, 1, 2, \ldots.$$

Answer: The process is irreducible because any state $k > 0$ communicates with state 0 because we have a positive probability for the following cycle:

$$0 \to 1 \to 2 \to \cdots \to k \to 0.$$

The transition probabilities for this Markov chain are

$$P_{i,i+1} = p, \quad P_{i,0} = 1 - p \quad \text{for all } i = 0, 1, 2 \ldots.$$

Other P_{ij}'s are all 0.

Since the process is irreducible, the limiting distribution is simply the solution to the balanced equation:

$$\pi_i = \sum_j \pi_j P_{ji} = p\pi_{i-1} = p^2 \pi_{i-2} = \cdots = p^i \pi_0.$$

To make $\sum_{i=0}^{\infty} \pi_i = 1$,

$$1 = \sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} p^i = \frac{\pi_0}{1-p},$$

we must have $\pi_0 = (1 - p)$. The limiting distribution is thus

$$\pi_i = \lim_{n \to \infty} \mathrm{P}(X_n = i) = (1-p)p^i \quad \text{for } i = 0, 1, 2, \ldots.$$

(c) [10 pts] Let T_k be the first time that k consecutive heads have appeared. In other words, $T_k = m$ if and only if at mth toss the Markov chain $\{X_n : n \geq 0\}$ reaches state k for the first time. Explain why that $\mathbb{E}[T_k]$'s satisfy the recursive equation

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1, \quad k = 2, 3, 4, \ldots$$

and also show that $\mathbb{E}[T_1] = 1/p$.

Answer: To reach state k from state 0, the process must first reach state $k-1$. It takes T_{k-1} steps to reach state $k-1$. If the in the next toss the coin comes up heads, then state k is reached and so we have $T_k = 1 + T_{k-1}$. However, if the next toss comes up tails, then the process goes back to state 0 and one need another T_k' steps to reach state k. Here T_k' has the same distribution as T_k. Moreover, T_k' is independent of the past, and hence is independent of T_{k-1} To sum up,

$$T_k = \begin{cases} T_{k-1} + 1 & \text{with probability } p, \\ T_{k-1} + 1 + T_k' & \text{with probability } 1-p. \end{cases}$$

So

$$\begin{aligned} \mathbb{E}[T_k|T_{k-1}] &= T_{k-1} + 1 + (1-p)\mathbb{E}[T_k'|T_{k-1}] \\ &= T_{k-1} + 1 + (1-p)\mathbb{E}[T_k'] & \text{(because } T_k' \text{ is independent of } T_{k-1}\text{)} \\ &= T_{k-1} + 1 + (1-p)\mathbb{E}[T_k] & \text{(because } T_k' \sim T_k\text{)} \end{aligned}$$

and hence

$$\mathbb{E}[T_k] = \mathbb{E}[\mathbb{E}[T_k|T_{k-1}]] = \mathbb{E}[T_{k-1}] + 1 + (1-p)\mathbb{E}[T_k]$$

Subtracting $(1-p)\mathbb{E}[T_k]$ from both sides of the equation we get

$$p\mathbb{E}[T_k] = \mathbb{E}[T_{k-1}] + 1, \quad k = 2, 3, 4, \ldots.$$

For T_1, observe that T_1 has a geometric distribution since $T_1 = n$ if and only if the coin comes up tails in the first $(n-1)$ tosses and heads in the nth toss. So

$$\mathrm{P}(T_1 = n) = (1-p)^{n-1}p \quad \text{for } n = 1, 2, 3, \ldots.$$

One can then show that $\mathbb{E}[T_1] = 1/p$.

(d) [6 pts] Solve the recursive equation in part (c) to find $\mathbb{E}[T_k]$.

Answer:

$$\begin{aligned}
\mathbb{E}[T_1] &= \frac{1}{p} \\
\mathbb{E}[T_2] &= (\mathbb{E}[T_1] + 1)/p = \frac{1}{p} + \frac{1}{p^2} \\
\mathbb{E}[T_3] &= (\mathbb{E}[T_2] + 1)/p = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \\
&\vdots
\end{aligned}$$

We can see that

$$\mathbb{E}[T_k] = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^k}.$$