Statistics 25300/31700
Introduction to Probability Models

Winter 2014 - Final Exam
Wednesday, March 19, 2014 4-6 PM

Student Name (print): ________________________________

Please Sign the Following Honor Statement.

I will not try to record, copy, or disclose any exam question or answer, in whole or in part, in any form or by any means (orally, in writing, on any internet chat room, or otherwise) to any students before 12:30pm, Friday, March 21, 2014.

(Sign Your Name Here)________________________________

(a) This is a closed-book, closed-note examination. You can refer to your two page of formula sheets.

(b) You can use a calculator.

(c) *You need to show your work to receive full credit. In particular, if you are basing your calculations on a formula or an expression (e.g., $E(Y|X = k)$), write down that formula before you substitute numbers into it.*

(d) If a later part of a question depends on an earlier part, the later part will be graded conditionally on how you answered the earlier part, so that a mistake on the earlier part will not cost you all points on the later part. *If you can't work out the actual answer to an earlier part, put down your best guess and proceed.*

(e) Do not pull the pages apart. If a page falls off, sign the page. If you do not have enough room for your work in the place provided, ask for extra papers, label and sign the pages.

(f) There is a normal table on p.14

	Question	*Points Available*	*Points Earned*
1	Batteries	40	
2	Brownian Motion	20	
3	Barber Shop	40	
	TOTAL	100	

Problem 1. Suppose a certain machine needs one battery to function. Whenever the battery in use fails, it is immediately replaced with a new one. Suppose that there are two brands of batteries, Brand X and Brand Y, and that for political reasons a company buys batteries of both types for replacement.

- When a Brand X battery fails it is replaced with a new Brand Y battery and when a Brand Y battery fails it is replaced with a new Brand X battery.
- The lifetimes (measured in thousands of hours) of Brand X batteries are uniform on $[1, 2]$ and the Brand Y batteries have lifetimes that are uniform on $[1, 3]$.
- The lifetime of any battery is independent of the lifetimes of other batteries.

Answer the following questions for large time t.

(a) [5 pts] What is the rate (how many batteries per 1000 hours) that batteries are replaced?

Answer: This is a alternating renewal process. We say the system is ON if the battery in use of Brand X and OFF if of Brand Y. The mean length of ON time is thus $(1+2)/2 = 1.5$, and OFF time is $(1 + 3)/2 = 2$. In one cycle (ON + OFF), the battery is replaced twice, so the rate batteries are replaced is

$$\frac{2}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{2}{1.5 + 2} = \frac{4}{7} \text{ battery per 1000 hours.}$$

(b) [5 pts] What is the long run proportion of time the system uses a battery of Brand X?

Answer: The long run proportion of time the system uses a battery of Brand X is

$$\frac{\mathbb{E}[\text{ON}]}{\mathbb{E}[\text{ON}] + \mathbb{E}[\text{OFF}]} = \frac{1.5}{1.5 + 2} = \frac{3}{7}.$$

(c) [5 pts] What is the long run proportion of time that the current battery in use is less than 1 thousand-hour old?

Answer: Let X_i be the lifetime of ith battery of brand X used, and Y_i be the lifetime of ith battery of brand Y used. So $X_i + Y_i$ is the length of the ith cycle.
We can view this as a renewal reward process, where the reward R_i in the ith cycle is the length of time that the current battery in use is less than 1 thousand-hour old. When the ith battery of brand X is in use, the length of time it is less than 1 thousand-hour old is $\min(X_i, 1)$. Similarly, the length of such time for the ith battery of brand Y is $\min(Y_i, 1)$. So the reward in the ith cycle is

$$R_i = \min(X_i, 1) + \min(Y_i, 1) = 1 + 1 = 2.$$

Here $R_i = 2$ because $X_i \sim$ Uniform(1,2)≥ 1 and $Y_i \sim$ Uniform(1,3) ≥ 1. So the proportion of time that the current battery in use is less than 1 thousand-hour old is simply the rate the reward accumulates,

$$\frac{\mathbb{E}[R_i]}{\mathbb{E}[X_i] + \mathbb{E}[Y_i]} = \frac{2}{1.5 + 2} = \frac{4}{7}.$$

For part (d)(e)(f)(g) below, instead of alternating the brands, the following battery replacement rule is used. Whenever the failed battery lasts more than 2 thousand hours, it is replaced with a battery of the same brand. Otherwise, it is replaced with a battery of the other brand.

(d) [5 pts] Suppose the system starts with a battery of brand Y. Let Y_i be the lifetime of the ith brand Y battery used and let N be the total number of brand Y batteries used before the first brand X battery is in use. So the length of time until the system starts to use the first brand X battery is

$$\sum_{i=1}^{N} Y_i.$$

Is N a *stopping time* for the sequence $\{Y_i : i = 1, 2, 3, \ldots\}$?

Answer: Yes, N a stopping time for the sequence $\{Y_i : i = 1, 2, 3, \ldots\}$ because the event $\{N = k\}$ is equivalent to the event

$$\{Y_i > 2 \text{ for } 1 \leq i \leq k-1, \text{ and } Y_k \leq 2\}$$

which only depends on $\{Y_i : 1 \leq i \leq k\}$ but not $\{Y_i : i > k\}$.

(e) [5 pts] (This continues part (d).) Given that $N = k$, find $\mathbb{E}\left[\sum_{i=1}^{N} Y_i \middle| N = k\right]$ and show that it is not equal to $k\mathbb{E}[Y_1]$.

Answer: Given $N = k$, $Y_i \sim$ Uniform(2,3) for $1 \leq j \leq k-1$ and $Y_k \sim$ Uniform(1,2). So $\mathbb{E}[Y_i|N = k] = 2.5$ for $i < k$ and $\mathbb{E}[Y_k|N = k] = 1.5$.

$$\begin{aligned}\mathbb{E}\Big[\sum_{i=1}^{N} Y_i \Big| N = k\Big] &= \Big(\sum_{i=1}^{k-1} \mathbb{E}[Y_i|N = k]\Big) + \mathbb{E}[Y_k|N = k] \\ &= 2.5 \times (k-1) + 1.5 = 2.5k - 1.\end{aligned}$$

which is not equal to $k\mathbb{E}[Y_1] = 2k$.

(f) [5 pts] Give a sequence of events in this battery replacement process that the number of such events occurred by time t constitute a (possibly delayed) renewal process and explain why.

Answer: For example, we say an event occurs whenever the system switches from a Brand Y to a Brand X battery. Within a cycle, the first battery is of Brand X and the second battery must be of Brand Y, since the lifetime of a Brand Y battery never exceeds 2 thousand hours. Then the machine continually uses Brand Y batteries until one of them lasts less than 2 thousand hours, and then the next event occurs that a Brand X battery is replaced.
Let X_i be the lifetime of the Brand X battery in the ith cycle. Let N_i be the number of Brand Y battery used in the ith cycle. Let $Y_{i,j}$ be the lifetime of the jth Brand Y battery used in the ith cycle. Observe that

$$\mathrm{P}(N_i = k) = \mathrm{P}(Y_{i,j} > 2, \text{ for } 1 \le j \le k-1, \text{ and } Y_{i,k} \le 2).$$

So the length of the ith cycle is

$$X_i + Y_{i,1} + \cdots + Y_{i,N_i}.$$

Since the lifetimes of batteries are independent, and N_i only depends on the lifetimes of batteries used within the cycle. So the interarrival times are independent. Moreover, since X_i's are uniform on (1,2), and all $Y_{i,j}$'s are uniform on (1,3), and N_i's are geometrically distributed because

$$\begin{aligned} \mathrm{P}(N_i = k) &= \mathrm{P}(Y_{i,j} > 2, \text{ for } 1 \le j \le k-1, \text{ and } Y_{i,k} \le 2) \\ &= [\textstyle\prod_{j=1}^{k-1} \mathrm{P}(Y_{i,j} > 2)]\mathrm{P}(Y_{i,k} \le 2) = (1/2)^k, \end{aligned}$$

we can see the interarrival times are identically distributed. So the events corresponds to the replacement of a Brand X battery indeed constitute a (delayed) renewal process.

(g) [10 pts] What is the rate that the machine replaces batteries (how many per 1000 hours)?

Answer: As in part (f), we say a cycle starts whenever the system switches from a brand Y battery to a brand X battery. The length of the ith cycle is $X_i + Y_{i,1} + \cdots + Y_{i,N_i}$. In the ith cycle, one brand X battery and N_i brand Y batteries are used. Consider this problem as a renewal reward process, where the reward is the number of batteries in the cycle. Observe that the rate batteries are replaced is simply

$$\frac{\text{mean reward in a cycle}}{\text{mean length of a cycle}} = \frac{1 + \mathbb{E}[N_i]}{\mathbb{E}[X_i] + \mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i}]}.$$

We know $\mathbb{E}[X_i] = 1.5$ since $X_i \sim$ Uniform(1,2). Note for all i, N_i distributed as the stopping time N in (d). From part (e) we know that

$$\mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i} | N_i] = 2.5 \times N + 1.5 = 2.5N - 1.$$

Since N_i is geometric with $\mathrm{P}(N_i = k) = (1/2)^k$, we know $\mathbb{E}[N_i] = 2$ and hence

$$\begin{aligned}\mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i}] &= \mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i} | N_i]] \\ &= \mathbb{E}[2.5N_i - 1] = 2.5\mathbb{E}[N_i] - 1 = 2.5 \times 2 - 1 = 4.\end{aligned}$$

Or alternatively, one may use the fact that N_i is a stopping time, so

$$\mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i}] = \mathbb{E}[N_i]\mathbb{E}[Y_{i,1}] = 2 \times 2 = 4.$$

So the rate that batteries are replaced is

$$\frac{1 + \mathbb{E}[N_i]}{\mathbb{E}[X_i] + \mathbb{E}[Y_{i,1} + \cdots + Y_{i,N_i}]} = \frac{1+2}{1.5+4} = \frac{3}{5.5} = \frac{6}{11} \text{ (batteries per 1000 hours).}$$

Problem 2. [Brownian Motion] **[20 points]**
The process $\{B(t) : t \geq 0\}$ is standard Brownian motion (with drift $\mu = 0$ and variance parameter $\sigma = 1$).

(a) [5 pts] What is $\mathbb{E}[B(1)B(3)|B(2) = 1]$?

Answer:

$$\begin{aligned}
\mathbb{E}[B(1)B(3)|B(1), B(2) = 1] &= B(1)\mathbb{E}[B(3)|B(2) = 1] \\
&= B(1)E[B(2) + B(3) - B(2)|B(2) = 1] \\
&= B(1)\Big(\underbrace{E[B(2)|B(2) = 1]}_{=1} + \underbrace{\mathbb{E}[B(3) - B(2)|B(2) = 1]}_{=\mathbb{E}[B(3)-B(2)]=0}\Big) \\
&= B(1)
\end{aligned}$$

in which $\mathbb{E}[B(3) - B(2)|B(2) = 1] = \mathbb{E}[B(3) - B(2)] = 0$ because $B(3) - B(2)$ is independent of $B(2)$ by the independent increment property of the Brownian motion

$$\begin{aligned}
\mathbb{E}[B(1)B(3)|B(2) = 1] &= \mathbb{E}[\underbrace{\mathbb{E}[B(1)B(3)|B(1), B(2) = 1]}_{B(1)}|B(2) = 1] \\
&= \mathbb{E}[B(1)|B(2) = 1] \\
&= 1/2.
\end{aligned}$$

Here we use the conditional distribution of $B(s)$ given $B(t) = x$ for $s < t$ derived in class, that

$$B(s)|_{B(t)=x} \sim N\left(\frac{s}{t}x, \frac{s(t-s)}{t}\right).$$

(b) [5 pts] Find the probability that a standard Brownian motion $\{B(t) : t \geq 0\}$ has ever hit -1 sometime before time $t = 1$?

Answer: The problem asks for $\mathrm{P}(\min_{0\leq t\leq 1} B(t) \leq -1)$. Observe the mirror image of a standard Brownian motion about the zero line $\{-B(t) : t \geq 0\}$ is also a standard Brownian motion. So

$$\mathrm{P}(\min_{0\leq t\leq 1} B(t) \leq -1) = \mathrm{P}(\max_{0\leq t\leq 1} B(t) \geq 1).$$

Recall that maximum of a standard Brownian motion $\max_{0\leq s\leq t} B(s)$ has the same distribution as $|B(t)|$.

$$\begin{aligned}\mathrm{P}\left(\max_{0\leq t\leq 1} B(t) > 1\right) &= 2\mathrm{P}(B(1) > 1) = 2 \times (1 - \Phi(1)) \\ &= 2 \times (1 - 0.8413) = 0.3174.\end{aligned}$$

(c) [10 pts] Find the probability $\mathrm{P}\left(\max_{0\le s\le 25} B(s) > 5,\ 0 < B(25) < 2\right)$.

Answer: Let $T_5 = \min\{t : B(t) = 5\}$ be the first time the Brownian motion hits value 5. By the Reflection principle, the reflected process

$$\overline{B}(t) = \begin{cases} B(t) & \text{if } t < T_5 \\ 10 - B(t) & \text{if } t \ge T_5 \end{cases}$$

is also a standard Brownian motion.

The event $\{\max_{0\le s\le 25} B(s) > 5\}$ means the Brownian motion has touched 5 before time $t = 25$, $T_5 < 25$, So

$$\overline{B}(25) = 10 - B(25).$$

Also observe the reflected process has touched 5 before time $t = 25$ if and only if the original process has, too. So the following two events are equivalent

$$\left\{\max_{0\le s\le 25} B(s) > 5,\ 0 < B(25) < 2\right\} = \left\{\max_{0\le s\le 25} \overline{B}(s) > 5,\ 0 < 10 - \overline{B}(25) < 2\right\}.$$

That $0 < 10 - \overline{B}(25) < 2$ means $\overline{B}(25)$ is between 8 and 10, which automatically implies $\max_{0\le s\le 25} \overline{B}(s) > 5$. So

$$\mathrm{P}\Big(\max_{0\le s\le 25} B(s) > 5, 0 < B(25) < 2\Big) = \mathrm{P}(8 < \overline{B}(25) < 10).$$

Since $\overline{B}(25) \sim N(0, 25)$, we have

$$\begin{aligned} \mathrm{P}(8 < \overline{B}(25) < 10) &= \mathrm{P}(\overline{B}(25) < 10) - \mathrm{P}(\overline{B}(25) < 8) \\ &= \Phi(10/\sqrt{25}) - \Phi(8/\sqrt{25}) \\ &= \Phi(2) - \Phi(1.6) \\ &\approx 0.9772 - 0.9452 = 0.0320. \end{aligned}$$

Problem 3. [Barber Shop] [**40 points**]
A barber shop has a mater barber and an apprentice barber.

- The two barbers each works on one customer at a time.
- The master barber can serve μ_M customers per hour and the apprentice barber can serve μ_A per hour. All service times are exponentially distributed, mutually independent, and is independent of the customer arrival process.
- Potential customers come to the barber shop according to a Poisson process at constant rate of λ per hour.
- An arrival finding the master barber free will begin service with the master barber. An arrival finding the master barber busy and the apprentice barber free will enter service with the apprentice barber. An arrival finding both servers busy goes away. Once a customer is served by either barber, he/she departs the system.

(a) [10 pts] Define states so as to be able to analyze this system. Give a set of linear equations whose solution will yield the long-run proportion of time the system is in each state.

No need to solve the equations.

Answer: Let $X(t)$ be a stochastic process with 4 states: $\{0, M, A, 2\}$, such that we say

$$X(t) = \begin{cases} 0 & \text{if no customer in the barber shop at time } t \\ M & \text{if the master barber is busy but the apprentice barber is free at time } t \\ A & \text{if the apprentice barber is busy but the master barber is free at time } t \\ 2 & \text{if both barbers are busy at time } t \end{cases}$$

The process $\{X(t), t \geq 0\}$ is a continuous-time Markov chain because the time between transitions are exponentially distributed and the transition rates are

$$\begin{aligned} &q_{0M} = \lambda, && \Rightarrow \nu_0 = q_{0M} = \lambda, \\ &q_{M0} = \mu_M,\ q_{M2} = \lambda, && \Rightarrow \nu_M = q_{M0} + q_{M2} = \mu_M + \lambda, \\ &q_{A0} = \mu_A,\ q_{A2} = \lambda, && \Rightarrow \nu_A = q_{A0} + q_{A2} = \mu_A + \lambda, \\ &q_{2M} = \mu_A,\ q_{2A} = \mu_M, && \Rightarrow \nu_2 = q_{2M} + q_{2A} = \mu_A + \mu_M. \end{aligned}$$

All other transition rates are 0.

Denote the steady state distribution as P_0, P_M, P_A, P_2. The set of balance equations $\mu_j P_j = \sum_i q_{ij} P_i$ is

$$\begin{aligned} \nu_0 P_0 &= q_{M0} P_M + q_{A0} P_A & \Rightarrow && \lambda P_0 &= \mu_M P_M + \mu_A P_A \\ \nu_A P_A &= q_{2A} P_2 & \Rightarrow && (\lambda + \mu_A) P_A &= \mu_M P_2 \\ \nu_M P_M &= q_{0M} P_0 + q_{2M} P_2 & \Rightarrow && (\lambda + \mu_M) P_M &= \lambda P_0 + \mu_A P_2 \\ \nu_2 P_2 &= q_{M2} P_M + q_{A2} P_A & \Rightarrow && (\mu_A + \mu_M) P_2 &= \lambda P_M + \lambda P_A \end{aligned}$$

along with $P_0 + P_M + P_A + P_2 = 1$.

Answer part (b)(c)(d)(e) assuming $\lambda = 4$, $\mu_M = 4$, and $\mu_A = 2$.

(b) [5 pts] Solve the system of equations for the steady-state distribution in part (a) assuming $\lambda = 4$, $\mu_M = 4$, $\mu_A = 2$.

Answer: The system of equations is

$$\begin{aligned}
\lambda P_0 &= \mu_M P_M + \mu_A P_A &&\Rightarrow & 4P_0 &= 4P_M + 2P_A \\
(\lambda + \mu_A) P_A &= \mu_M P_2 &&\Rightarrow & 6P_A &= 4P_2 \\
(\lambda + \mu_M) P_M &= \lambda P_0 + \mu_A P_2 &&\Rightarrow & 8P_M &= 4P_0 + 2P_2 \\
(\mu_A + \mu_M) P_2 &= \lambda P_M + \lambda P_A &&\Rightarrow & 6P_2 &= 4P_M + 4P_A.
\end{aligned}$$

along with $P_0 + P_M + P_A + P_2 = 1$. The solution is

$$(P_0, P_M, P_A, P_2) = \left(\frac{7}{22}, \frac{5}{22}, \frac{4}{22}, \frac{6}{22}\right)$$

(c) [5 pts] What proportion of entering customers are served by the master barber?

Answer: An customer finds the barber shop in state 0, A, or M at arrival will enter the shop. And those who arrive when the barber shop is in state 0 or A will be serve by the master barber. So the proportion of entering customers served by the master barber is

$$\frac{P_0 + P_A}{P_0 + P_M + P_A} = \frac{7 + 4}{7 + 5 + 4} = \frac{11}{16}.$$

(d) [10 pts] What is the average time an entering customer spends in the system?

Answer: Based on the answers in part (b), we know the average number of customers in the barber is:

$$L = 1 \times P_M + 1 \times P_A + 2 \times P_2 = 1 \times \frac{5}{22} + 1 \times \frac{4}{22} + 2 \times \frac{6}{22} = \frac{21}{22} \approx 0.9545.$$

Since not all potential customers enter the shop, the rate that customers enter the system is not $\lambda = 4$, but

$$\lambda_a = 4(P_0 + P_A + P_M) = 4\left(\frac{7+5+4}{22}\right) = \frac{64}{22}.$$

Using cost identity $L = \lambda_a W$ in chapter 8, we can get the average amount of time customers spent in the is

$$W = L/\lambda_a = \frac{21/22}{64/22} = \frac{21}{64} \approx 0.328 \text{ hour.}$$

Another argument is the following. If a customer finds that barber shop in state 0 or A at arrival, he/she will be serve by the master barber, then his/her mean service time will be $1/\mu_M = 1/4$ hour. If a customer finds that barber shop in state M at arrival, he/she will be serve by the apprentice barber, then his/her mean service time will be $1/\mu_A = 1/2$ hour.

Note that no customer will enter the system when that barber shop in state 2.

Thus the average of time customers spent in the system is:

$$\frac{P_0 \times (1/4) + P_A \times (1/4) + P_M \times (1/2)}{P_0 + P_A + P_M} = \frac{21}{64} =\approx 0.328 \text{ hour.}$$

(e) [10 pts] Suppose the apprentice barber just finished serving one customer. How long is he expected to wait until he becomes busy (begins to serve another customer) again.

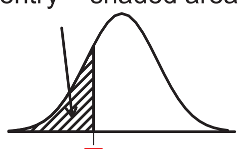
Answer: We can view the system as a alternating renewal process that the apprentice barber alternates between two states: busy and idle. Using the theory of alternating renewal process, and the result in part(b), the proportion of time that the apprentice barber is busy is

$$\frac{\mathbb{E}[\text{busy}]}{\mathbb{E}[\text{busy}] + \mathbb{E}[\text{idle}]} = P_A + P_2 = \frac{4+6}{22} = \frac{10}{22}.$$

Since the system has no queue, the length of a busy period is simply the time to serve one customer, which is, $\mathbb{E}[\text{busy}] = 1/\mu_A = 1/2$ hour, from which we can get that

$$\frac{1/2}{1/2 + \mathbb{E}[\text{idle}]} = \frac{10}{22} \quad \Rightarrow \quad \mathbb{E}[\text{idle}] = 3/5 = 0.6 \text{ hour.}$$

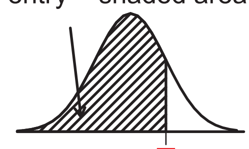
table entry = shaded area



Standard Normal Probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

table entry = shaded area



Standard Normal Probabilities

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998