8.7 The Model G/M/1

The G/M/1 model assumes:
- i.i.d times between successive arrivals with an arbitrary distribution G
- i.i.d service times $\sim \text{Exp}(\mu)$
- a single server; and
- first come, first serve

Just like M/G/1 system, there is also a discrete-time Markov chain embedded in an G/M/1 system. Let $Y_n = \#$ of customers in the system seen by the $n$th arrival, $n \geq 1$

$A_n = \#$ of customers the server can possibly serve between the $(n-1)$st and the $n$th arrival, $n \geq 1$

Observed that $\{Y_n, n \geq 0\}$ and $\{A_n, n \geq 1\}$ are related as follows

$$Y_{n+1} = \begin{cases} Y_n + 1 - A_{n+1} & \text{if } Y_n + 1 \geq A_{n+1} \\ 0 & \text{if } Y_n + 1 < A_{n+1} \end{cases}, \quad n \geq 1$$

A Markov Chain embedded in G/M/1 (Cont’d)

- By the memoryless property of the exponential service time, the remaining service time of the customer being served at an arrival is also $\sim \text{Exp}(\mu)$.
- Thus starting from the $(n-1)$st arrival, the events of completion of servicing a customer constitute a Poisson process of rate $\mu$.
- Let $G_n$ be the time elapsed between the $(n-1)$st and the $n$th arrival.
- Let $G_n$, $A_n$ be Poisson with mean $\mu G_n$.
- As $G_n$’s are i.i.d $\sim G$, we can conclude that $A_1, A_2, \ldots$ are i.i.d. with distribution

$$\alpha_k = P(A_n = k) = \int_0^\infty P(A_n = k | G_n = y) G(dy)$$

$$= \int_0^\infty \frac{(\mu y)^k}{k!} e^{-\mu y} G(dy)$$

A Markov Chain embedded in G/M/1 (Cont’d)

The transition probabilities $P_{ij}$ for the Markov chain $\{Y_n, n \geq 0\}$ are thus:

$$P_{ij} = P(Y_{n+1} = j | Y_n = i) = \begin{cases} P(A_{n+1} \geq i + 1) = \sum_{k=i+1}^\infty \alpha_k & \text{if } j = 0 \\ P(A_{n+1} = i + 1 - j) = \alpha_{i+1-j} & \text{if } j \geq 1, i + 1 \geq j \\ 0 & \text{if } i + 1 < j \end{cases}$$

i.e., the transition probability matrix is

$$P = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & \cdots \\ \sum_{k=1}^\infty \alpha_k & \alpha_0 & 0 & 0 & 0 & \cdots \\ \sum_{k=2}^\infty \alpha_k & \alpha_1 & \alpha_0 & 0 & 0 & \cdots \\ \sum_{k=3}^\infty \alpha_k & \alpha_2 & \alpha_1 & \alpha_0 & 0 & \cdots \\ \sum_{k=4}^\infty \alpha_k & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
A Markov Chain embedded in $G/M/1$ (Cont’d)

To find the stationary distribution $\pi_j = \lim_{n \to \infty} P(Y_n = i),\ i = 0, 1, 2, \ldots$, we have to solve the equations

$$\pi_j = \sum_{i=0}^{\infty} \pi_i P_{ij} = \sum_{i=0}^{\infty} \pi_i \alpha_{i+1-j} \beta, \ j \geq 1 \quad \text{and} \quad \sum_{j=0}^{\infty} \pi_j = 1$$

Let us try a solutions of the form $\pi_j = c \beta^j, j \geq 0$. Substitution into the equation above leads to

$$c \beta^j = \sum_{i=j-1}^{\infty} c \beta^i \alpha_{i+1-j} \Rightarrow \beta = \sum_{i=j-1}^{\infty} \beta^i \alpha_{i+1-j} = \sum_{i=0}^{\infty} \beta^i \alpha_i$$

Observe that $\sum_{i=0}^{\infty} \beta^i \alpha_i$ is exactly the generating function of $A_n$ $G(s) = \mathbb{E}[s^{A_n}]$ taking value at $s = \beta$.

Thus if we can find $0 < \beta < 1$ such that $\beta = g(\beta)$, then

$$\pi_j = (1 - \beta) \beta^j, \ j \geq 0$$

is a stationary distribution of $\{Y_n\}$.

PASTA Principle Does Not Apply to $G/M/1$

With the stationary distribution $\{\pi_j, j \geq 0\}$, one might attempt to calculate $L$, the average number of customers in the system as

$$\mathbb{E}[Y_n] = \sum_{k=0}^{\infty} \pi_k = \sum_{k=0}^{\infty} k(1 - \beta) \beta^k = \frac{\beta}{1 - \beta}.$$ 

However, the PASTA principle does not apply as the arrival process is not Poisson. Recall

$$\pi_k = \text{proportion of customers see } k \text{ in the system at arrival}$$

$$P_k = \text{proportion of time having } k \text{ customers in the system},$$

Observe that $\pi_k \neq P_k$ since the longer the interarrival time $G_n$, the larger $A_n$, and hence the smaller $Y_n$. Hence

$$L = \sum_{k=0}^{\infty} P_k \neq \mathbb{E}[Y_n] = \sum_{k=0}^{\infty} \pi_k.$$

$W$ of $G/M/1$

Though we cannot use $\{\pi_j\}$ to find $L$, we can use it to find $W$.

Let $W_n$ be the waiting time of $n$th customer in the system. If he/she see $k$ customers at arrival, then $W_n$ is the total service time of $k + 1$ customers. That is,

$$\mathbb{E}[W_n|Y_n = k] = \mathbb{E}[\text{sum of } k + 1 \text{ i.i.d. } \text{Exp}(\mu) \text{ service times}] = \frac{k + 1}{\mu}.$$

Thus

$$W = \sum_{k=0}^{\infty} \mathbb{E}[W_n|Y_n = k] P(Y_n = k) = \sum_{k=0}^{\infty} \mathbb{E}[W_n|Y_n = k] \pi_k$$

$$= \sum_{k=0}^{\infty} \frac{k + 1}{\mu} (1 - \beta) \beta^k = \frac{1}{\mu(1 - \beta)} \frac{1}{(1 - \beta)^2}. $$

Here we use the identity $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1 - x)^2}$. 

A Markov Chain embedded in $G/M/1$ (Cont’d)

The equation

$$\beta = g(\beta)$$

has a solution between 0 and 1 iff $g'(1) = \mathbb{E}[A_n] = \mu \mathbb{E}[G_n] > 1$

since

This condition is intuitive since if

the average service time $1/\mu$

> the average interarrival time of customers $\mathbb{E}[G_n]$,

the queue will become longer and longer and the system will ultimately explode.
By the Little’s Formula, we know $L = \lambda W$, in which $\lambda$ is the arrival rate of customers, which is the reciprocal of the mean interarrival time $E[G_n]$

$$\lambda = \frac{1}{E[G_n]}$$

Thus

$$L = \lambda W = \frac{1}{E[G_n]} \frac{1}{\mu(1-\beta)} = \frac{1}{\mu E[G_n](1-\beta)}$$

Moreover,

$$W_Q = W - E[Service \ Time] = W - \frac{1}{\mu} = \frac{\beta}{\mu(1-\beta)}$$

$$L_Q = \lambda W_Q = \frac{\beta}{\mu E[G_n](1-\beta)}$$

8.9.3 $G/M/k$

Just like $G/M/1$ system, $G/M/k$ system can also be analyzed as a Markov Chain. Let

- $Y_n = \#$ of customers in the system seen by the $n$th arrival, $n \geq 1$
- $A_n = \#$ of customers the $k$ servers can possibly serve between the $(n-1)$st and the $n$th arrival, $n \geq 1$

Observed again that $\{Y_n, n \geq 0\}$ and $\{A_n, n \geq 1\}$ are related as follows

$$Y_{n+1} = \begin{cases} Y_n + 1 - A_{n+1} & \text{if } Y_n + 1 \geq A_{n+1} \\ 0 & \text{if } Y_n + 1 < A_{n+1} \end{cases}, \quad n \geq 1$$

One can show that the distribution of $A_{n+1}$ depends on $Y_n$ but not $Y_{n-1}, Y_{n-2}, \ldots$ and hence $\{Y_n\}$ is a Markov chain. The transition probabilities are given in [IPM10e] p.565-566.

8.9.4 $M/G/k$

Unlike $G/M/k$, the method to analyze $M/G/1$ cannot be used to analyze $M/G/k$. If we follow the lines as we do in $M/G/1$

- $Y_n = \#$ of customers in the system leaving behind at the $n$th departure, $n \geq 1$
- $A_n = \#$ of customers entered the system during the service time of the $n$th customer, $n \geq 1$

As there are more than one server, the service times are not disjoint, and hence $A_n$’s are not independent.

In fact, there is NO known exact formula for $L$, $W$, $L_Q$, $W_Q$ of an $M/G/k$ system.

10.6 Gaussian Processes

**Definition 10.2**

A stochastic process $\{X(t), t \geq 0\}$ is called a Gaussian process if $X(t_1), \ldots, X(t_n)$ has a multivariate normal distribution for all $t_1, \ldots, t_n$.

Because a multivariate normal distribution is completely determined by the marginal mean values and the covariance values it follows that the properties of a Gaussian process is completely determined by its **mean function**

$$m(t) = E[X(t)]$$

and **covariance function**

$$C(s, t) = Cov(X(s), X(t))$$

That is, two Gaussian process are the same if their **mean functions** and **covariance functions** are identical.
Definition of a Brownian Motion

**Definition 1** A stochastic process \( \{B(t), t \geq 0\} \) is said to be a Brownian motion if:

(i) \( B(0) = 0 \);

(ii) \( \{B(t), t \geq 0\} \) has stationary and independent increments;

(iii) for every \( t, s > 0 \), \( B(t + s) - B(s) \sim N(0, \sigma^2 t) \)

A Brownian motion with \( \sigma = 1 \) is called a standard Brownian motion process.

In fact, we can show that, as a function of \( t \), the path of \( B(t) \) is continuous w/ prob. 1.

Covariance Function of a Brownian Motion

For \( t > s \)

\[ \text{Cov}[B(t), B(s)] = \text{Cov}[B(t) - B(s) + B(s), B(s)] \\
= \text{Cov}[B(t) - B(s), B(s)] + \text{Cov}[B(s), B(s)] \\
= 0 + \text{Var}[B(s)] \quad \text{(by independent increment)} \\
= \sigma^2 s \]

Thus

\[ \text{Cov}(B(t), B(s)) = \sigma^2 \min(s, t) \]

\( \{B(t), t \geq 0\} \) is a Brownian motion.

Alternatively, a Brownian motion can be defined as a Gaussian process with mean function

\[ m(t) = \mathbb{E}[B(t)] = 0 \]

and covariance function

\[ C(s, t) = \text{Cov}(B(s), B(t)) = \sigma^2 \min(s, t). \]

Properties of a Brownian Motion

Let \( \{B(t), t \geq 0\} \) be a standard Brownian motion, then each of the following process is also a standard Brownian motion:

(i) \( \{-B(t), t \geq 0\} \)

(ii) \( \{B(t + s) - B(s), t \geq 0\} \)

(iii) \( \{aB(t/a^2), t \geq 0\} \)

(iv) \( \{tB(1/t), t \geq 0\} \)

**Pf.** We’ll prove (iv) only. The rest of the proof are similar. Clearly \( \{tB(1/t), t \geq 0\} \) is a Gaussian process since it is a linear function of a Brownian motion process.

\[
\mathbb{E}[tB(1/t)] = t\mathbb{E}[B(1/t)] = 0 \quad \text{since} \quad B(1/t) \sim N(0, 1/t)
\]

\[
\text{Cov}[tB(1/t), sB(1/s)] = ts\text{Cov}[B(1/t), B(1/s)]
\]

\[
= ts \min\left(\frac{1}{t}, \frac{1}{s}\right) = \begin{cases} 
    ts(1/t) = s & \text{if } t > s \\
    ts(1/s) = t & \text{if } t \leq s
\end{cases}
\]

\[
= \min(s, t)
\]

As the Gaussian process \( \{tB(1/t), t \geq 0\} \) has the same mean function and variance function as a standard Brownian motion, it is also a standard Brownian motion.