Problems for Turn In:

1. Recall the Branching process we discussed in class and in Section 4.7 of [IPM10e]. Let \( X_n \) denote the size of the \( n \)-th generation, \( n = 0, 1, 2, \ldots \), and let \( Z_{n,i} \) be the number of offsprings produced by the \( i \)-th individual at the end of the \( n \)-th generation. Suppose \( Z_{n,i} \)’s are i.i.d. with distribution

\[
P(Z_{n,i} = j) = P_j, \quad j \geq 0,
\]

for all \( n, i \). We suppose that \( P_j < 1 \) for all \( j \geq 0 \). The size of the \((n+1)\)-th generation will then be

\[
X_{n+1} = \sum_{i=1}^{X_n} Z_{n,i}
\]

Let \( g(s) = \mathbb{E}[s^{Z_{n,i}}] = \sum_{k=0}^{\infty} P_k s^k \) be the generating function of \( Z_{n,i} \), and \( G_n(s) \) be the generating function of \( X_n, n = 0, 1, 2, \ldots \). Show that if \( X_0 = 1 \),

\[
G_{n+1}(s) = g(G_n(s)).
\]

Hint: Since \( X_0 = 1 \), observe that \( X_{n+1} \) is the sum of \( X_1 \) independent copies of \( X_n \) (why?). Find \( \mathbb{E}[s^{X_{n+1}}|X_1] \) first and then \( G_{n+1}(s) = \mathbb{E}[s^{X_{n+1}}] = \mathbb{E}[\mathbb{E}[s^{X_{n+1}}|X_1]] \).

2. Recall in Example 3 of Lecture 1 we have introduced a discrete queueing process. The system consists of a single server and a queue with unlimited capacity. Suppose time is discretized and customers arrive only at the beginning of a time period. Let \( \xi_n \) be the number of customers arrived at the beginning of the \( n \)-th period. Suppose \( \{\xi_n, n = 0, 1, 2, \ldots \} \) are i.i.d. with

\[
P(\xi_n = k) = a_k, \quad k = 0, 1, 2, \ldots \quad a_k \geq 0 \text{ for all } k, \quad \text{and } \sum_{k=0}^{\infty} a_k = 1
\]

A customer arriving for service begins being served immediately if the the server is free and the queue is empty upon arrival; otherwise, the customer joins the end of the queue. If two or more customers arrive at the same time, then one of them will go into service immediately if the server is free and the queue is empty, while the other customers will go to the end of the queue. The customers are served in a first-come, first-served manner, so any customers in the queue will always be served before any newly arriving customers. In each time period the server begins serving (and completes the service of) exactly one customer if there are any to be served (from the queue if it is nonempty or a newly arriving customer if the queue is empty); otherwise, the server is idle for that time period. Let \( X_n \) be the number of customers in the system, including the one being served, at the \textit{end} of the \( n \)-th time period. In class we have argued that

\[
X_{n+1} = \begin{cases} X_n - 1 + \xi_{n+1} & \text{if } X_n \geq 1 \\ \xi_{n+1} & \text{if } X_n = 0 \end{cases}
\]

Let \( g(s) = \mathbb{E}[s^{\xi_n}] = \sum_{k=0}^{\infty} a_k s^k \) be the common generating function of all \( \xi_n \)’s.
(a) Let $G_n(s) = E[s^{X_n}]$ be the generating function of $X_n$. Show that

$$G_{n+1}(s) = g(s) \left( G_n(0) + \frac{G_n(s) - G_n(0)}{s} \right)$$

(b) Suppose $X_0 = 0$, and the distribution of $\xi_n$’s is

$$P(\xi_n = 0) = 1/2, \ P(\xi_n = 1) = P(\xi_n = 2) = 1/4, \text{ and } P(\xi_n > 2) = 0.$$  
That is, $a_0 = 1/2, a_1 = 1/4, a_2 = 1/4, \text{ and } a_k = 0 \text{ for } k = 3, 4, 5 \ldots$. Find the distribution of $X_2$.

(c) Assume that a stationary distribution $\pi = (\pi_0, \pi_1, \pi_2, \ldots)$ of $\{X_n\}$ exists (it does when $\mu = E(\xi_n) < 1$). That is, $P(X_n = k) = \pi_k$ for all $n \geq 0$. Let $G(s)$ be the generating function of this stationary distribution

$$G(s) = E(s^{X_n}) = \sum_{k=0}^{\infty} \pi_k s^k.$$  
Show that

$$G(s) = (1 - \mu) \frac{(s - 1)g(s)}{s - g(s)} \quad \text{where } \mu = g'(1) = E(\xi_n).$$

Hint: Explain why $G_{n+1}(s) = G(s)$ if $G_n(s) = G(s)$ and then use the recursive equation in (a). Show that $G(0) = 1 - \mu$ by letting $s \uparrow 1$.

(d) Find the stationary distribution of $\{X_n, n \geq 0\}$ when the distribution of $\xi_n$’s is as described in (b).

(e) Use part (c) to show that the mean of the stationary distribution $\lim_{n \to \infty} E[X_n]$ is

$$G'(1) = \mu + \frac{g''(1)}{2(1 - \mu)} = \mu + \frac{E[\xi_n(\xi_n - 1)]}{2(1 - \mu)}.$$