STAT 224 Lecture 8
Polynomial Models

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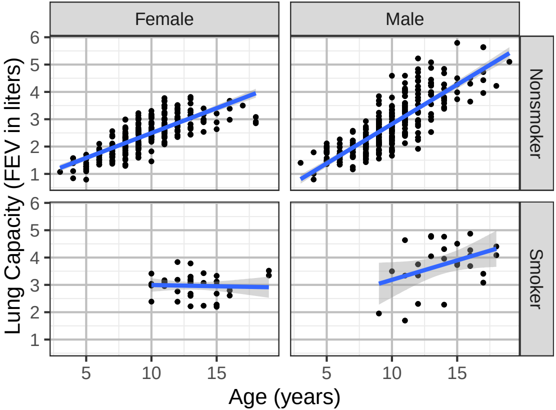
Does FEV Grow Linearly in Age?

Children stop growing after they turn adults.

FEV might not grow linearly with age, at least for **female nonsmokers** in the plots below.



Test of non-linearity (Female Nonsmokers)

Let's focus on female nonsmokers first.

```
f.nonsmokers = subset(fevdata, sex == "Female" & smoke == "Nonsmoker")
```

To test non-linearity, ane can add some nonlinear function of `age`, e.g. `age^2` and see if the nonlinear term is significant.

```
summary(lm(fev ~ age + I(age^2), data=f.nonsmokers))$coef
             Estimate Std. Error t value  Pr(>|t|)
(Intercept) -0.50746    0.21039  -2.412 1.652e-02
age          0.43979    0.04297  10.235 4.715e-21
I(age^2)    -0.01298    0.00212  -6.121 3.176e-09
```

- The tiny p-value for the `age^2` is strong evidence of non-linearity.

Polynomial Models

- Fitting the polynomial model

$$\text{fev} = \beta_0 + \beta_1 \text{age} + \beta_2 (\text{age})^2 + \varepsilon$$

doesn't means we believe it's correct. It might just be a decent approximation to the true underlying nonlinear model

$$\text{fev} = f(\text{age}) + \varepsilon$$

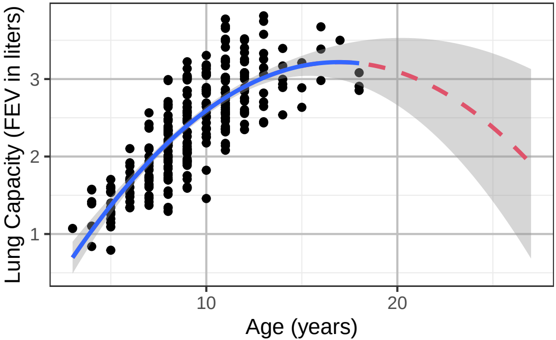
- One can try higher-order polymials

$$\text{fev} = \beta_0 + \beta_1 \text{age} + \beta_2 (\text{age})^2 + \cdots + \beta_k (\text{age})^k + \varepsilon$$

if lower-order ones don't capture the nonlinear pattern well.

Extrapolation is Dangerous!

Lung capacity decreases after children turn aduts?



We are not sure whether the nonlinear relations is a polynomial (it's just an approximation!). Extrapolating the model beyond the range of data is dangerous.

Test of Non-linearity (Male Nonsmokers)

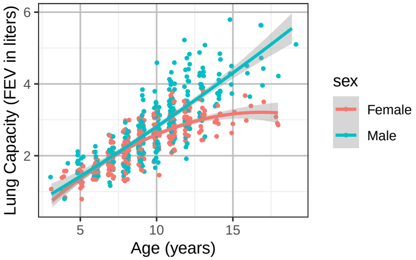
```
m.nonsmokers = subset(fevdata, sex == "Male" & smoke == "Nonsmoker")
summary(lm(fev ~ age + I(age^2), data=m.nonsmokers))$coef
            Estimate Std. Error t value   Pr(>|t|)
(Intercept) 0.143622   0.298683  0.4809 0.63096439
age         0.245875   0.059137  4.1577 0.00004175
I(age^2)    0.002058   0.002821  0.7296 0.46619995
```

- The large P-value 0.466 for the `age^2` means little evidence non-linearity
- This just means `fev` is approx. linear in `age` in the range of data for male smokers. Extrapolating the line beyond of the range of data remain dangerous
- The discrepancy in the significance of `age^2` between boys and girls is an evidence of `age:sex` interaction — lung capacities of girls stop growing earlier than boys.

Age:Sex Interactions — Nonsmokers Only

```
nonsmokers = subset(fevdata, smoke == "Nonsmoker")
summary(lm(fev ~ (age + I(age^2))*sex, data=nonsmokers))$coef
                 Estimate Std. Error t value  Pr(>|t|)
(Intercept)      -0.50746   0.263274  -1.927 5.440e-02
age               0.43979   0.053770   8.179 1.796e-15
I(age^2)         -0.01298   0.002653  -4.892 1.295e-06
sexMale           0.65108   0.369659   1.761 7.871e-02
age:sexMale      -0.19392   0.074369  -2.607 9.355e-03
I(age^2):sexMale  0.01504   0.003612   4.163 3.611e-05
```

Not surprisingly, the tiny P-value 3.61×10^{-5} for the term `I(age^2):sexMale` indicates that boys and girls differ significantly in the curvatures of the growth curves of lung capacities.



```
  (Intercept)               age         I(age^2)          sexMale
     -0.50746           0.43979         -0.01298          0.65108
  age:sexMale I(age^2):sexMale
     -0.19392           0.01504
```

$$\widehat{\text{fev}} \approx -0.507 + 0.44\text{Age} - 0.013(\text{Age})^2 + 0.651\text{Sex}_M$$
$$- 0.194(\text{Sex}_M \cdot \text{Age}) + 0.015(\text{Sex}_M \cdot \text{Age}^2)$$

The estimated growth curve for girls ($\text{Sex}_M = 0$) is

$$\widehat{\text{fev}} \approx -0.507 + 0.44\text{Age} - 0.013(\text{Age})^2,$$

The estimated growth curve for boys ($\text{Sex}_M = 1$) is

$$\widehat{\text{fev}} \approx (-0.507 + 0.651) + (0.44 - 0.194)\text{Age} + (-0.013 + 0.015)(\text{Age})^2$$
$$= 0.144 + 0.246\text{Age} + 0.002(\text{Age})^2$$

Observe the coefficients are identical to the coefficients for the model including female nonsmokers only and the one for male nonsmokers only.

Interpretation of Coefficients in a Polynomial Model

Recall in Lecture 3 we said β_j = the regression coefficient for X_j, is the mean change in the response Y when X_j is increased by one unit **holding other X_i's constant**.

However in a model that involves polynomial terms like

$$Y = \beta_0 + \underbrace{\beta_1 X_1 + \beta_2 X_1^2}_{\text{a polynomial of } X_1} + \beta_3 X_3 + \cdots + \beta_p X_p + +\varepsilon$$

it makes no sense to interpret a single coefficient for a polynomial like β_1 or β_2 since it's impossible to change X_1 while holding X_1^2 constant. We should interpret the entire polynomial altogether like,

> *the mean of Y change with X_1 following the curve $\beta_1 X_1 + \beta_2 X_1^2$* ***holding other X_i's constant****.*

Interpretation of Coefficients of Indicator Variables.

Ex: for the salary survey data, it's **incorrect** to interpret δ_2 in the model below

$$S = \beta_0 + \alpha M_1 + \delta_2 E_2 + \delta_3 E_3 + \beta X + \varepsilon$$

as

> *the mean change in salary S when E_2 is increased by 1 holding M_1, E_3 and X constant*

What's wrong?

- When E_2 increases from 0 to 1, E_3 might decrease from 1 to 0 or stay at 0, and hence we might not be able to hold E_3 constant.
- There is only one way E_2 can increase by 1 — from a HS diploma to a college degree. One should always interpret **in context** whenever possible.

The better interpretation for δ_2 would be:

> *the mean difference in salary S between HS graduates and those w/ a Bachelor's degree if they were at the same management status and had the same years of experience.*

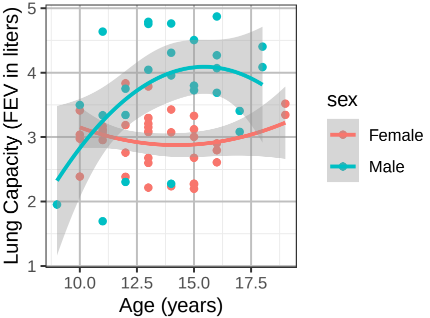
Test of Non-linearity (Smokers)

```
m.smokers = subset(fevdata, sex == "Male" & smoke == "Smoker")
summary(lm(fev ~ age + I(age^2), data=m.smokers))$coef
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -6.05764     4.7793  -1.267  0.21767
age          1.31395     0.7056   1.862  0.07539
I(age^2)    -0.04253     0.0255  -1.668  0.10889
f.smokers = subset(fevdata, sex == "Female" & smoke == "Smoker")
summary(lm(fev ~ age + I(age^2), data=f.smokers))$coef
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.98970    1.95968   3.056 0.004204
age         -0.43746    0.28431  -1.539 0.132627
I(age^2)     0.01536    0.01013   1.516 0.138246
```

- `age^2` is insignificant for male smokers or female smokers, might be just due to the small sample size that makes it difficult to detect the non-linearity.

```r
smokers = subset(fevdata, smoke == "Smoker")
summary(lm(fev ~ (age + I(age^2))*sex, data=smokers))$coef
                   Estimate Std. Error t value Pr(>|t|)
(Intercept)         5.98970    2.79719   2.141  0.03639
age                -0.43746    0.40582  -1.078  0.28543
I(age^2)            0.01536    0.01447   1.062  0.29252
sexMale           -12.04734    4.53162  -2.659  0.01009
age:sexMale         1.75141    0.66463   2.635  0.01073
I(age^2):sexMale   -0.05790    0.02390  -2.423  0.01850
```

Male smokers still have significantly larger lung capacities than female nonsmokers, though neither show significant non-linearity



Can we remove the square term `age^2` for smokers?

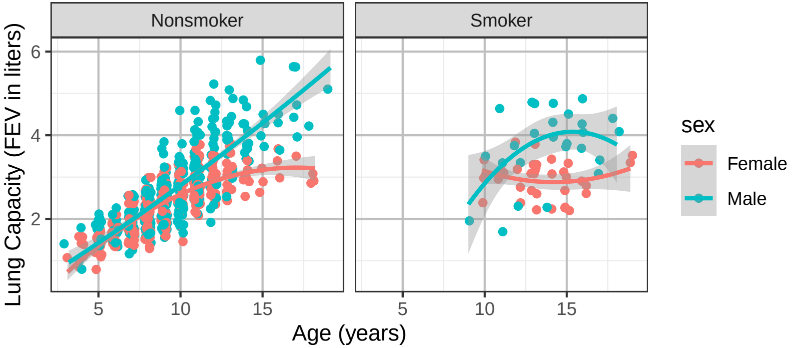
```
anova(lm(fev ~ (age + I(age^2))*sex, data=smokers),
        lm(fev ~ age*sex, data=smokers))
Analysis of Variance Table

Model 1: fev ~ (age + I(age^2)) * sex
Model 2: fev ~ age * sex
  Res.Df  RSS Df Sum of Sq    F Pr(>F)
1     59 21.3
2     61 23.5 -2     -2.21 3.06  0.054
```

The P-value 0.054 is at the borderline of significance, some moderate but not compelling evidence of non-linearity.

Nonlinear 3-Way Interaction Model

```
summary(lm(log(fev) ~ (age + I(age^2))*sex*smoke, data=fevdata))$coef
                              Estimate Std. Error t value  Pr(>|t|)
(Intercept)                  -0.665172   0.101526  -6.552 1.167e-10
age                           0.238379   0.020735  11.496 6.048e-28
I(age^2)                     -0.007802   0.001023  -7.625 8.822e-14
sexMale                       0.308447   0.142551   2.164 3.085e-02
smokeSmoker                   2.831422   0.873835   3.240 1.256e-03
age:sexMale                  -0.073525   0.028679  -2.564 1.058e-02
I(age^2):sexMale              0.004883   0.001393   3.506 4.864e-04
age:smokeSmoker              -0.395307   0.127614  -3.098 2.036e-03
I(age^2):smokeSmoker          0.013287   0.004604   2.886 4.031e-03
sexMale:smokeSmoker          -4.350760   1.413287  -3.078 2.169e-03
age:sexMale:smokeSmoker       0.651102   0.208206   3.127 1.845e-03
I(age^2):sexMale:smokeSmoker -0.023863   0.007545  -3.163 1.637e-03
```



Can you explain from the plot why the 3-way interaction term `I(age^2):sexMale:smokeSmoker` is significant?