# STAT 224 Lecture 3
# Multiple Linear Regression, Part 2

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# Outline

- Example: The Trees Data
- Standard Errors & Distributions of Least Squares Estimates for Coefficients
- Hypothesis Tests & Confidence Intervals for Coefficients

# Example: The Trees Data

# Example: The Trees Data

The `trees` data are measurements of the diameter, height and volume of timber in 31 felled black cherry trees. The variables are

- `Girth`: Tree diameter (rather than girth, actually) in inches measured at 4 ft 6 in above the ground
- `Height`: Height in ft
- `Volume`: Volume of timber in cubic ft

The `trees` data are build-in in R. One can load the the data by the command

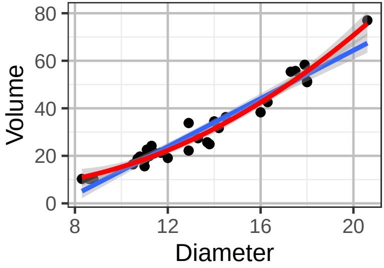
```
data("trees")
```

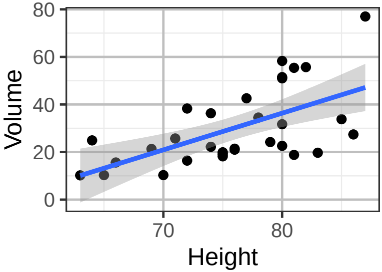
Let's rename the misleading `Girth` variable as `Diameter`

```
trees$Diameter = trees$Girth
```

# Pairwise Scatter Plots

```
library(ggplot2)
ggplot(trees, aes(x=Diameter, y=Volume)) + geom_point() +
  geom_smooth(method='lm', formula='y~x') +
  geom_smooth(method='lm', formula='y~x+I(x^2)', col="red")
ggplot(trees, aes(x=Height, y=Volume)) + geom_point() +
  geom_smooth(method='lm', formula='y~x')
```

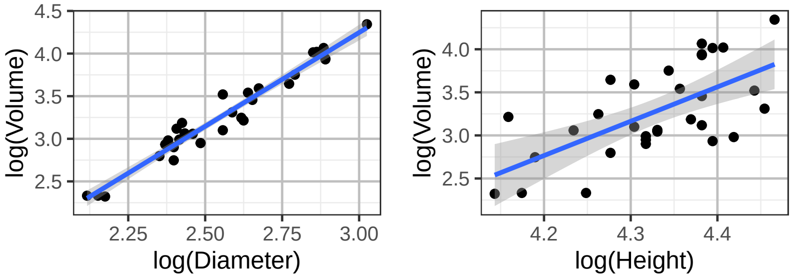


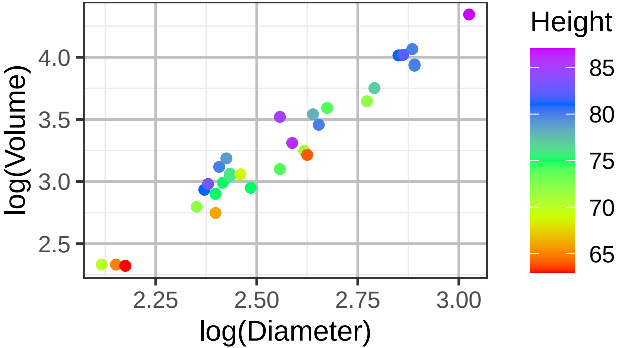

- slight *non-linearity* between `Diameter` & `Volume`
- Variability of `Volume` increases w/ `Height`

# After Log-Transformation . . .

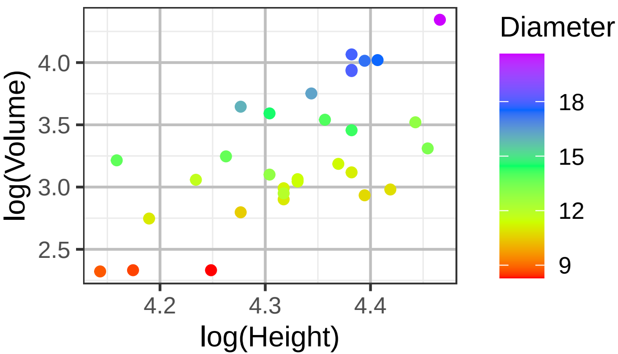
```r
ggplot(trees, aes(x=log(Diameter), y=log(Volume))) + geom_point() +
  geom_smooth(method='lm', formula='y~x')
ggplot(trees, aes(x=log(Height), y=log(Volume))) + geom_point() +
  geom_smooth(method='lm', formula='y~x')
```



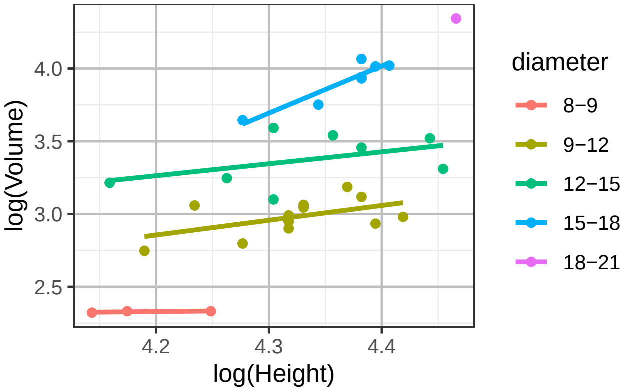
```r
ggplot(trees, aes(x=log(Diameter), y=log(Volume), color=Height)) +
  geom_point() + scale_color_gradientn(colours = rainbow(5))
```



```r
ggplot(trees, aes(x=log(Height), y=log(Volume), color=Diameter)) +
  geom_point() + scale_color_gradientn(colours = rainbow(5))
```



```r
trees$diameter = cut(trees$Diameter, breaks=c(8,9,12,15,18,21),
                     labels=c("8-9","9-12", "12-15", "15-18","18-21"))
ggplot(trees, aes(x=log(Height), y=log(Volume), color=diameter)) +
  geom_point() + geom_smooth(method='lm', formula='y~x', se=F)
```



# Model for the Timber Volume of Trees

Recall in the previous lecture we argued that

$$\text{Timber Volume} \approx (\text{constant})(\text{Diameter})^2(\text{Height})$$

Taking logarithm on both sides, we consider the model

$$\log(\text{Volume}) = \beta_0 + \beta_1 \log(\text{Diameter}) + \beta_2 \log(\text{Height}) + \varepsilon$$

and we expect $\beta_1 = 2$ and $\beta_2 = 1$. We thus fit the model

```
lmtrees = lm(log(Volume) ~ log(Diameter) + log(Height), data=trees)
lmtrees$coef
  (Intercept) log(Diameter)   log(Height)
       -6.632         1.983         1.117
```

We get $\widehat{\beta}_1 = 1.983$ and $\widehat{\beta}_2 = 1.117$.

Are they close to $\beta_1 = 2$ and $\beta_2 = 1$?

Need to know the *variability* of the LS estimates.

# Standard Errors & Distributions of Least Squares Estimates

# Least Squares Estimates Are Unbiased

Recall in L02, we said the LS estimate $\widehat{\beta}$ in matrix notation is

$$\widehat{\beta} = (X^T X)^{-1} X^T Y$$

Based on the MLR model in matrix notation $Y = X\beta + \varepsilon$, the expected value of $Y$ is

$$\mathrm{E}[Y] = \mathrm{E}[X\beta] + \underbrace{\mathrm{E}[\varepsilon]}_{=0} = X\beta$$

Recall in MLR, $X$ are regard as *fixed numbers*, no randomness. The expected value of $\widehat{\beta}$ is hence

$$\mathrm{E}[\widehat{\beta}] = \mathrm{E}[(X^T X)^{-1} X^T Y] = (X^T X)^{-1} X^T \underbrace{\mathrm{E}[Y]}_{=X\beta} = (X^T X)^{-1} X^T X\beta = \beta$$

The LS estimate $\widehat{\beta}$ is hence an *unbiased* estimate for $\beta$.

# Variance of the LS Estimate $\widehat{\beta}$

It can be shown that the variance of $\widehat{\beta}$ in matrix notation is

$$\mathrm{Var}(\widehat{\beta}) = \sigma^2 (X^T X)^{-1},$$

where

$$\mathrm{Var}(\widehat{\beta}) = \begin{bmatrix} \mathrm{Var}(\widehat{\beta}_0) & \mathrm{Cov}(\widehat{\beta}_0, \widehat{\beta}_1) & \mathrm{Cov}(\widehat{\beta}_0, \widehat{\beta}_2) & \cdots & \mathrm{Cov}(\widehat{\beta}_0, \widehat{\beta}_p) \\ \mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_0) & \mathrm{Var}(\widehat{\beta}_1) & \mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_2) & \cdots & \mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_p) \\ \mathrm{Cov}(\widehat{\beta}_2, \widehat{\beta}_0) & \mathrm{Cov}(\widehat{\beta}_2, \widehat{\beta}_1) & \mathrm{Var}(\widehat{\beta}_2) & \cdots & \mathrm{Cov}(\widehat{\beta}_2, \widehat{\beta}_p) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(\widehat{\beta}_p, \widehat{\beta}_0) & \mathrm{Cov}(\widehat{\beta}_p, \widehat{\beta}_1) & \mathrm{Cov}(\widehat{\beta}_p, \widehat{\beta}_2) & \cdots & \mathrm{Var}(\widehat{\beta}_p) \end{bmatrix}$$

and

$$(X^T X)^{-1} = \begin{bmatrix} n & \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i2} & \cdots & \sum_{i=1}^n x_{ip} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1} x_{i2} & \cdots & \sum_{i=1}^n x_{i1} x_{ip} \\ \sum_{i=1}^n x_{i2} & \sum_{i=1}^n x_{i2} x_{i1} & \sum_{i=1}^n x_{i2}^2 & \cdots & \sum_{i=1}^n x_{i2} x_{ip} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^n x_{ip} & \sum_{i=1}^n x_{ip} x_{i1} & \sum_{i=1}^n x_{ip} x_{i2} & \cdots & \sum_{i=1}^n x_{ip}^2 \end{bmatrix}^{-1}$$

# Variance of the LS Estimates for SLR

For SLR, $(X^T X)^{-1}$ equals

$$\begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix}^{-1} = \begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^n x_i^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} & -\frac{\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ -\frac{\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} & \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix}$$

Hence,

$$\begin{bmatrix} \mathrm{Var}(\widehat{\beta}_0) & \mathrm{Cov}(\widehat{\beta}_0, \widehat{\beta}_1) \\ \mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_0) & \mathrm{Var}(\widehat{\beta}_1) \end{bmatrix} = \sigma^2 (X^T X)^{-1} = \sigma^2 \begin{bmatrix} \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} & -\frac{\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} \\ -\frac{\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} & \frac{1}{\sum_{i=1}^n (x_i - \bar{x})^2} \end{bmatrix}$$

We get that

$$\mathrm{Var}(\widehat{\beta}_0) = \sigma^2 \left( \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2} \right), \quad \mathrm{Var}(\widehat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^n (x_i - \bar{x})^2},$$

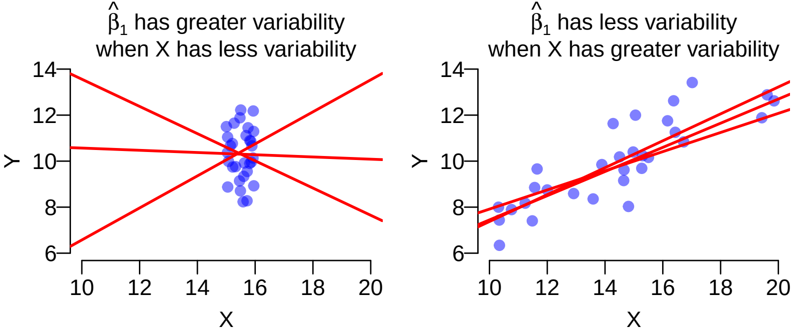
$$\mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_0) = -\frac{\sigma^2 \bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

# Greater Variability in $X$, Better Estimate for Slope (SLR)

$$\text{SD}(\widehat{\beta}_1) = \sqrt{\text{Var}(\widehat{\beta}_1)} = \frac{\sigma}{\sqrt{\sum_{i=1}^n (x_i - \overline{x})^2}} = \frac{\sigma}{\sqrt{n-1}\,\text{SD}_x}.$$

$\widehat{\beta}_1$ will be closer to $\beta_1$ if

1) the sample size $n$ is larger, or 2) $X$ has greater variability



**Remark 1**: Another way to derive $\mathrm{Var}(\widehat{\beta}_0)$:

$$\begin{aligned}\mathrm{Var}(\widehat{\beta}_0) = \mathrm{Var}(\bar{y} - \widehat{\beta}_1 \bar{x}) &= \mathrm{Var}(\bar{y}) - 2\bar{x}\,\overbrace{\mathrm{Cov}(\bar{y}, \widehat{\beta}_1)}^{=0} + \bar{x}^2\,\mathrm{Var}(\widehat{\beta}_1) \\ &= \frac{\sigma^2}{n} + 0 + \bar{x}^2 \frac{\sigma^2}{\sum_{i=1}^n (x_i - \overline{x})^2}\end{aligned}$$

- $\bar{y}$ and $\widehat{\beta}_1$ are uncorrelated because the slope ($\widehat{\beta}_1$) is invariant if you shift the response up or down ($\bar{y}$).

**Remark 2**: The LS estimates for the *slope* and the *intercept* are *negatively correlated*

$$\mathrm{Cov}(\widehat{\beta}_1, \widehat{\beta}_0) = \mathrm{E}[(\widehat{\beta}_1 - \beta_1)(\widehat{\beta}_0 - \beta_0)] = \frac{-\sigma^2\,\bar{x}}{\sum_{i=1}^n (x_i - \overline{x})^2}$$

- Usually, if the slope estimate is too high, the intercept estimate is too low

# Standard Error (s.e.) of the LS Estimate

For MLR in general,

- $\text{Var}(\widehat{\beta}_j) = \sigma^2 \times$(the $j$th diagonal element of $(X^T X)^{-1}$), $j = 0, 1, \ldots, p$
- $\text{Cov}(\widehat{\beta}_j, \widehat{\beta}_k) = \sigma^2 \times$(the $(j, k)$ entry of $(X^T X)^{-1}$), $j, k = 0, 1, \ldots, p$
- $s.e.(\widehat{\beta}_j) = \sqrt{\text{Var}(\widehat{\beta}_j)}$ but the unknown $\sigma^2$ is replaced by MSE.

For SLR

$$s.e.(\widehat{\beta}_1) = \sqrt{\frac{\text{MSE}}{\sum_{i=1}^n (x_i - \bar{x})^2}}, \quad s.e.(\widehat{\beta}_0) = \sqrt{\text{MSE}} \sqrt{\frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}$$

# R Can Compute the Standard Errors!

Don't worry about the computation of $s.e.(\widehat{\beta}_j)$.

*R can compute the **standard errors**!*

```
lmtrees = lm(log(Volume) ~ log(Diameter) + log(Height), data=trees)
summary(lmtrees)$coef
              Estimate Std. Error t value  Pr(>|t|)
(Intercept)     -6.632    0.79979  -8.292 5.057e-09
log(Diameter)    1.983    0.07501  26.432 2.423e-21
log(Height)      1.117    0.20444   5.464 7.805e-06
```

- The column "**estimate**" shows the LS estimates

$$\widehat{\beta}_0 = -6.6316, \quad \widehat{\beta}_1 = 1.9826, \quad \text{and} \quad \widehat{\beta}_2 = 1.1171$$

- The column "**std. error**" gives the standard errors:

$$s.e.(\widehat{\beta}_0) = 0.7998, \quad s.e.(\widehat{\beta}_1) = 0.075 \quad \text{and} \quad s.e.(\widehat{\beta}_2) = 0.2044$$

```r
lmtrees = lm(log(Volume) ~ log(Diameter) + log(Height), data=trees)
summary(lmtrees)

Call:
lm(formula = log(Volume) ~ log(Diameter) + log(Height), data = trees)

Residuals:
     Min       1Q   Median       3Q      Max 
-0.16856 -0.04849  0.00243  0.06364  0.12922 

Coefficients:
              Estimate Std. Error t value     Pr(>|t|)
(Intercept)     -6.632      0.800   -8.29 0.0000000051
log(Diameter)    1.983      0.075   26.43      < 2e-16
log(Height)      1.117      0.204    5.46 0.0000078053

Residual standard error: 0.0814 on 28 degrees of freedom
Multiple R-squared:  0.978,	Adjusted R-squared:  0.976 
F-statistic:  613 on 2 and 28 DF,  p-value: <2e-16
```

# Hypothesis Tests & Confidence Intervals for Coefficients

# t-Test of a Single $\beta_j$

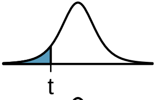
For an MLR model $Y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_p X_{ip} + \varepsilon_i$, the $t$-statistic for testing H$_0$: $\beta_j = \beta_j^0$ is

$$t = \frac{\widehat{\beta}_j - \beta_j^0}{s.e.(\widehat{\beta}_j)} \quad \text{which has a } t\text{-distribution with df} = n - p - 1$$

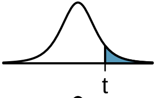
where $s.e.(\widehat{\beta}_j)$ is given on the previous slide.

The P-value can be calculated using `pt()` based on the alternative hypothesis H$_1$.

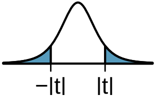
$P$-value =




= pt(t, df = n-p-1) if H$_1$: $\beta_j < \beta_j^0$




= pt(t, df = n-p-1, lower.tail=F) if H$_1$: $\beta_j > \beta_j^0$




= 2*pt(abs(t), df = n-p-1, lower.tail=F) if H$_1$: $\beta_j \neq \beta_j^0$

```
summary(lmtrees)$coef
               Estimate Std. Error t value  Pr(>|t|)
(Intercept)      -6.632    0.79979  -8.292 5.057e-09
log(Diameter)     1.983    0.07501  26.432 2.423e-21
log(Height)       1.117    0.20444   5.464 7.805e-06
```

As we believe $\beta_1 = 2$ and $\beta_2 = 1$, can test them w/ the $t$-statistics

$$t_1 = \frac{\widehat{\beta}_1 - 2}{s.e.(\widehat{\beta}_1)} = \frac{1.9826 - 2}{0.075} \approx -0.2313 \text{ with df } = 31 - 2 - 1 = 28$$

$$t_2 = \frac{\widehat{\beta}_2 - 1}{s.e.(\widehat{\beta}_2)} = \frac{1.1171 - 1}{0.2044} \approx 0.5729 \text{ with df } = 31 - 2 - 1 = 28.$$

The two-sided $p$-values are about 0.82 and 0.57

```
2*pt(0.2313, df = 28, lower.tail=F)
[1] 0.8188
2*pt(0.5729, df = 28, lower.tail=F)
[1] 0.5713
```

That is, “Volume $\approx$ (Diameter)$^2$(Height)” seems reasonable.

```r
summary(lmtrees)$coef
              Estimate Std. Error t value  Pr(>|t|)
(Intercept)     -6.632    0.79979  -8.292 5.057e-09
log(Diameter)    1.983    0.07501  26.432 2.423e-21
log(Height)      1.117    0.20444   5.464 7.805e-06
```

- The column "**statistic** shows the t-statistic for testing $\mathrm{H}_0$: $\beta_j = 0$ against $\mathrm{H}_1$: $\beta_j \neq 0$,

$$t_0 = \frac{\widehat{\beta_0} - 0}{s.e.(\widehat{\beta_0})} = \frac{-6.632 - 0}{0.79979} = -8.292,$$

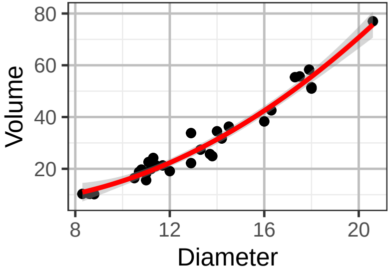
$$t_1 = \frac{\widehat{\beta_1} - 0}{s.e.(\widehat{\beta_1})} = \frac{1.983 - 0}{0.07501} = 26.432,$$

$$t_2 = \frac{\widehat{\beta_2} - 0}{s.e.(\widehat{\beta_2})} = \frac{1.117 - 0}{0.20444} = 5.464$$

  which are simply **the ratios of the first two columns**
- The column *Pr(> |t|)* shows the **2-sided** $P$-values for testing $\mathrm{H}_0$: $\beta_0 = 0$ and $\mathrm{H}_0$: $\beta_1 = 0$.

# Digression: Checking Non-Linearity

Recall we said earlier that the relation between Volume and Diameter is *slightly nonlinear*. We can check **nonlinearity** by fitting the polynomial model




$$\text{Volume} = \beta_0 + \beta_1 \text{Diameter} + \beta_2 (\text{Diameter})^2 + \varepsilon$$

and test $H_0$: $\beta_2 = 0$

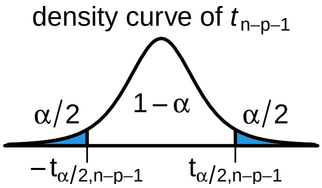
```
lm2 = lm(Volume ~ Diameter + I(Diameter^2), data=trees)
summary(lm2)$coef
                Estimate Std. Error t value  Pr(>|t|)
(Intercept)      10.7863   11.22282  0.9611 0.3447282
Diameter         -2.0921    1.64734 -1.2700 0.2145344
I(Diameter^2)     0.2545    0.05817  4.3756 0.0001524
```

# Confidence Intervals For Coefficients

The $100(1-\alpha)\%$ confidence interval for $\beta_j$ is

$$\widehat{\beta}_j \pm t_{(n-p-1,\alpha/2)} s.e.(\widehat{\beta}_j)$$

where $t_{(n-p-1,\alpha/2)} = t^*$ is the critical value for the $t_{n-p-1}$ distribution at confidence level $1-\alpha$, i.e.,




which can be found using either of the R commands

```
qt(alpha/2, df=n-p-1, lower.tail=FALSE)
qt(1-alpha/2, df=n-p-1)
```

# Example: CI for $\beta_1$

```
summary(lmtrees)$coef
              Estimate Std. Error t value  Pr(>|t|)
(Intercept)     -6.632    0.79979  -8.292 5.057e-09
log(Diameter)    1.983    0.07501  26.432 2.423e-21
log(Height)      1.117    0.20444   5.464 7.805e-06
```

A 95% confidence interval for $\beta_1$ is

$$\begin{aligned}\widehat{\beta}_1 \pm t_{0.05/2,28}\mathrm{SE}(\widehat{\beta}_1) &\approx 1.983 \pm 2.048 \times 0.07501 \\ &\approx 1.983 \pm 0.1536 \approx (1.829, 2.137)\end{aligned}$$

where $t_{0.05/2,28} \approx 2.048$ can be found using either R commands below

```
qt(0.05/2, df=28, lower.tail=F)
[1] 2.048
qt(0.975, df=28)
[1] 2.048
```

## Finding CIs for Coefficients Using `confint()`

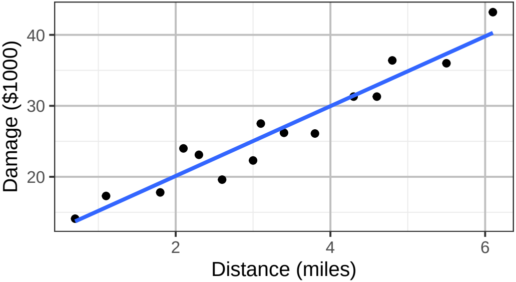
The `confint()` command in R can produce confidence intervals for the coefficients $\beta_0$ and $\beta_1$ for us

```
confint(lmtrees)
                 2.5 % 97.5 %
(Intercept)   -8.2699 -4.993
log(Diameter)  1.8290  2.136
log(Height)    0.6984  1.536
confint(lmtrees, level = 0.9)  # changing the confidence level to 90%
                  5 %   95 %
(Intercept)   -7.9922 -5.271
log(Diameter)  1.8550  2.110
log(Height)    0.7693  1.465
confint(lmtrees, level = 0.95, "log(Diameter)")
               2.5 % 97.5 %
log(Diameter) 1.829  2.136
confint(lmtrees, level = 0.95, "(Intercept)")
             2.5 % 97.5 %
(Intercept) -8.27 -4.993
```

# Example: Fire Damage Data

| Distance (mile) | Damage ($1000) |
|---|---|
| 0.7 | 14.1 |
| 1.1 | 17.3 |
| 1.8 | 17.8 |
| 2.1 | 24.0 |
| 2.3 | 23.1 |
| 2.6 | 19.6 |
| 3.0 | 22.3 |
| 3.1 | 27.5 |
| 3.4 | 26.2 |
| 3.8 | 26.1 |
| 4.3 | 31.3 |
| 4.6 | 31.3 |
| 4.8 | 36.4 |
| 5.5 | 36.0 |
| 6.1 | 43.2 |

An insurance company wanted to relate the amount of fire damage in major residential fires to the distance between the burning house and the nearest fire station. A sample of 15 recent fires was selected in a large suburb of a major city.



## Fire Damage Data

```
fire = data.frame(
  dist=c(0.7,1.1,1.8,2.1,2.3,2.6,3.0,3.1,3.4,3.8,4.3,4.6,4.8,5.5,6.1),
  damage=c(14.1,17.3,17.8,24.0,23.1,19.6,22.3,27.5,26.2,26.1,31.3,
           31.3,36.4,36.0,43.2)
  )
lmfire = lm(damage ~ dist, data=fire)
lmfire$coef
(Intercept)        dist
     10.278       4.919
```

$\widehat{\text{fire damage in \$1000}} = 10.28 + 4.92 \times (\text{distance to the nearest fire dept})$

- The intercept 10.278 means that the predicted amount of fire damage for houses located right next to a fire station is $10,278.
- The slope 4.919 means that every extra mile from the nearest fire station increases the amount of fire damage by $4,919 on average

## Example: Test for the Slope $\beta_1$

| | Estimate | Std.Error | t value | Pr(>\|t\|) |
|---|---|---|---|---|
| (Intercept) | 10.2779 | 1.4203 | 7.237 | 6.59e-06 |
| dist | 4.9193 | 0.3927 | 12.525 | 1.25e-08 |

To test $\mathrm{H}_0$: $\beta_1 = 4$ v.s. $\mathrm{H}_A$: $\beta_1 > 4$, the $t$-statistic is

$$t = \frac{b_1 - 4}{\mathrm{SE}(b_1)} = \frac{4.9193 - 4}{0.3927} = 2.3409, \quad \text{df} = n - 2 = 15 - 2 = 13.$$

The upper one-sided $P$-value can be found in R to be $\approx 0.018$.

```r
pt(2.3409, df=13, lower.tail=F)
[1] 0.01791
```

Conclusion: At 5% level, the extra amount of damage for every extra mile from the nearest fire station is significantly higher than $4000 on average.

# Example: CI for $\beta_1$

| | Estimate | Std.Error | t value | Pr(>\|t\|) |
|---|---|---|---|---|
| (Intercept) | 10.2779 | 1.4203 | 7.237 | 6.59e-06 |
| dist | 4.9193 | 0.3927 | 12.525 | 1.25e-08 |

A 95% confidence interval for $\beta_1$ is

$$\widehat{\beta}_1 \pm t_{0.05/2,13}\text{SE}(\widehat{\beta}_1) \approx 4.9193 \pm 2.16 \times 0.3927$$
$$\approx 4.919 \pm 0.848 \approx (4.071, 5.767)$$

where $t_{0.05/2,13} \approx 2.16$ can be found by he R command

```
qt(0.05/2, df=13, lower.tail=F)
[1] 2.16
confint(lmfire, "dist")
     2.5 % 97.5 %
dist 4.071  5.768
```

Interpretation: We have 95% confidence that every extra mile from the nearest fire station increases the amount of damage by $4071 to $5767 on average

# Coming Up Next

- Confidence Intervals and Prediction Intervals for Prediction
- Sum of Squares
- Model Comparison