STAT22000 Autumn 2013 Lecture 11

Yibi Huang

October 23, 2013

- Conditional Probability
- General Multiplication Rule
- Independence of Events
- The Rule of Total Probability
- Bayes' Rule

Textbook Coverage: Section 4.2 and 4.5

Conditional Probabilities

Example: You are drawing cards from a "perfectly" shuffled deck.

- What is the probability that the first card drawn is an ace?

$$P(\text{1st card is an ace}) = \frac{4}{52} = \frac{1}{13}.$$

- What is the probability that the 2nd card drawn is an ace when the first card drawn was unknown?

$$P(\text{2nd card is an ace}) = \frac{1}{13}$$

- What is the probability that the second card is an ace if the first card was known to be an ace? $\frac{3}{51}$

Conditional Probabilities

Given two events A and B. We denote the probability of event A happens **given** that event B is known to happen as

$$P(A|B),$$

read as the probability of "A given B."

For the example on the previous slide, let

$$A = \text{ 1st card is an ace,}$$
$$B = \text{ 2nd card is an ace.}$$

We have

$$P(B|A) = \frac{3}{51} \neq P(B) = \frac{4}{52}.$$

Current knowledge (outcome of the first draw) has changed (restricted) the sample space (possible outcomes) for future events.

General Multiplication Rule (1)

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that both cards are aces?

- Imagine maace many such deals.
  - The 1st card will be an ace about 4/52 of the time.
  - Among the deals where the 1st card is an ace, the 2nd card will be an ace about 3/51 of the time.
  - So both cards will be aces about 4/52 of 3/51 of the time.
- The probability that both cards are aces equals:

(The *unconditional* probability that the 1st card is an ace)

× (the *conditional* probability that the 2nd card is an ace *given* that the 1st card is an ace)

$$= \frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

General Multiplication Rule (2)

(The probability that two things will both happen)

= (the unconditional probability that the 1st will happen)

× (the conditional probability that the 2nd will happen given that the 1st has happened).

In mathematical notation,

$$\boxed{P(A \text{ and } B) = P(A) \times P(B|A)}$$

General multiplication Rule for several events:

$$P(ABCD) = P(A) \times P(B|A) \times P(C|AB) \times P(D|ABC)$$

and so on

An Example for the General Multiplication Rule

A deck of cards is shuffled and the two top cards are placed face down on a table. What is the probability that neither card is an ace?

Solution. Let

$$A = \text{ 1st card is NOT an ace,}$$
$$B = \text{ 2nd card is NOT an ace.}$$

- $P(A) = P(\text{the 1st card is not a ace}) =$ 48/52 .
- Given that the 1st card is not a ace, the conditional probability that the 2nd card is not a ace =? $P(B|A) = \frac{47}{51}$.
- So the probability that both cards are not aces =?

$$P(A \text{ and } B) = P(A) \times P(B|A) = \frac{48}{52} \times \frac{47}{51} = \frac{188}{221} \approx 0.851.$$

An Alternative Way to Find Conditional Probability

In view of the general multiplication rule

$$P(A \text{ and } B) = P(A) \times P(B|A),$$

we sometimes compute the conditional probability $P(B|A)$ via the formula

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

when $P(A \text{ and } B)$ and $P(A)$ are easier to find.

Remark: The formula above is sometimes adopted as the definition of conditional probability.

See the next page for an example.

Example — Formula 1 Race (1)

Let A be the winning team in a Formula 1 race: Red Bull, McLaren or Ferrari. Let B be the track condition: either *dry* or *wet*.

Winning Team	**Condition** *Dry*	*Wet*	$P(A)$
Red Bull	0.36	0.025	0.385
McLaren	0.27	0.025	0.295
Ferrari	0.27	0.050	0.320
$P(B)$	0.900	0.10	1.000

- Each cell gives the probability $P(A \text{ and } B)$ for a particular combination of a team and a condition.
- The probabilities of the 6 cells add up to 1 because we enumerate all possibilities (in this simplified Formula 1).

Example — Formula 1 Race (2)

Now let's condition on the event B. The track is either dry, which occurs with probability $P(B = Dry) = 0.9$, or wet, which occurs with probability $P(B = Wet) = 0.1$.

Team	**Conditional** $P(A|Dry)$	$P(A|Wet)$	**Unconditional** P(A)
Red Bull	$\frac{0.36}{0.9} = 0.40$	$\frac{0.025}{0.1} = 0.25$	0.385
McLaren	$\frac{0.27}{0.9} = 0.30$	$\frac{0.025}{0.1} = 0.25$	0.295
Ferrari	$\frac{0.27}{0.9} = 0.30$	$\frac{0.005}{0.1} = 0.50$	0.320
Total	1.00	1.00	

- Red Bull has higher probability to win on dry track; Ferrari has higher probability to win on wet track
- $\sum_A P(A|Dry) = 1$ because conditioning implies normalizing: by definition $P(A|B) = P(A \text{ and } B)/P(B)$.

Example — College Students (Ex. 4.44 on the Textbook)

age	full-time	part-time
15 to 19	0.21	0.03
20 to 24	0.32	0.07
25 to 34	0.10	0.10
35+	0.05	0.13

- Each cell gives the probability $P(A \text{ and } B)$ for a combination of full/part-time and age groups.
- What is the probability that a student is enrolled full-time? $P(\text{full time}) = 0.21 + 0.32 + 0.10 + 0.13 = 0.76.$
- What is the probability that a full-time student is between 25 and 34 years of age? $P(\text{age 25-34}|\text{full time}) = \frac{P(\text{full time and 25-34})}{P(\text{full time})} = \frac{0.1}{0.76} \approx 0.132.$
- What is the probability that a student who is between 25 and 34 years of age is enrolled full-time?

$$P(\text{full time }|\text{age 25-34}) = \frac{P(\text{full time and 25-34})}{P(\text{age 25-34})} = \frac{0.1}{0.1 + 0.1} = 0.5.$$

Independence

Two events A and B are **independent** if the probability for B given A are the same, no matter where A are true or not. Otherwise, they are **dependent**.

In mathematical notation,

$$A \text{ and } B \text{ are independent if } P(B|A) = P(B)$$

Example: Someone is going to roll a die twice. Are the two rolls independent, or dependent?

- No matter how the 1st roll turns out, the 2nd roll will give 1, 2, 3, 4, 5, or 6, with equal probabilities. So the two rolls are independent .

An Example of Dependent Events

A deck of cards is shuffled and the two top cards are placed face down on a table.

- Event A: the 1st card is a ace.
- Event B: the 2nd card is a ace.

Q: Are these two events independent, or dependent?
A:

- Given that the 1st card is a ace, the probability that the 2nd card is a ace equals $P(B|A) = \frac{3}{51}$.
- If the 1st card is unknown, the probability that the 2nd card is a ace equals $P(B) = 4/52$.

The probabilities for the 2nd event change, depending on how the 1st event turns out. So the two events are dependent.

Multiplication Rule for Independent Events

By the general multiplication rule,

$$P(A \text{ and } B) = P(A) \times P(B|A)$$

when A and B are independent, then $P(B|A) = P(B)$. Hence, we have

$$P(A \text{ and } B) = P(A) \times P(B)$$

In general, if several events are independent,
the probability that all of them will happen
equals the product of their unconditional probabilities.

Alternative Definitions of Independence

Two events A and B are **independent** if any of the following ones is true

- $P(B|A) = P(B)$
- $P(B|A) = P(B|A^c)$
- $P(AB) = P(A)P(B)$

Example of Multiplication Rules for Independent Events

Every day you buy a lottery ticket that offers 1 probability in 1000 of winning. What is the probability that you never win in 1000 plays?

The question asks for the probability of losing on each play.

- The plays are independent.
- Your probability of losing on any particular play = 0.999.
- Your probability of losing on all 1000 plays = $(0.999)^{1000}$, or 0.368 .

The probability that you win at least once in 1000 plays equals $1 - 0.368$, or 0.632 .

- The complement rule is useful here.

The Rule of Total Probability

Suppose the events $A_1, \ldots, A_k$ form a **partition** of the sample space S in which A_i's form a **partition** means they are

- mutually exclusive, i.e., $A_i \cap A_j = \emptyset$ whenever $i \neq j$;
- exhaustive, i.e. $A_1 \cup \cdots \cup A_k = S$ and

$$P(A_1) + \cdots + P(A_k) = 1$$

Then

$$\begin{aligned} P(B) &= P(B \text{ and } A_1) + \ldots + P(B \text{ and } A_k) \\ &= P(B|A_1)P(A_1) + \ldots + P(B|A_k)P(A_k) \end{aligned}$$

Example for the Rule of Total Probability

Suppose an applicant for a job has been invited for an interview.

The probability that

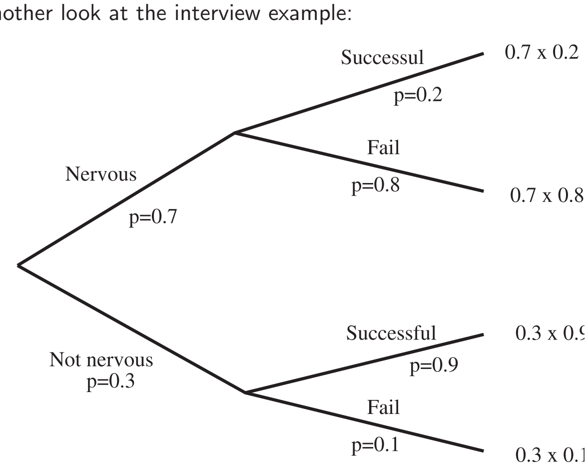
- he is nervous is $P(N) = 0.7$,
- the interview is successful when he is nervous is $P(S|N) = 0.2$,
- the interview is successful when he is not nervous is $P(S|N^c) = 0.9$.

What is the probability that the interview is successful?

$$\begin{aligned} P(S) &= P(S \text{ and } N) + P(S \text{ and } N^c) \\ &= P(S|N)P(N) + P(S|N^c)P(N^c) \\ &= 0.2 \times 0.7 + 0.9 \times 0.3 = 0.41 \end{aligned}$$

Tree Diagram for the Rule of Total Probability

Another look at the interview example:



Interview Example Continued

Conversely, given the interview is successful, what is the probability that the job applicant is nervous during the interview?

$$\begin{aligned} P(\text{Nervous}|\text{Successful}) &= \frac{P(\text{Nervous and Successful})}{P(\text{Successful})} \\ &= \frac{P(\text{Nervous and Successful})}{0.41} \\ &= \frac{P(\text{Successful}|\text{Nervous})P(\text{Nervous})}{0.41} \\ &= \frac{0.2 \times 0.7}{0.41} = \frac{14}{41} \approx 0.34. \end{aligned}$$

in which $P(\text{Successful}) = 0.41$ was found in the previous slide.

Bayes' Rule

The problem in the previous slide is an example of the **Bayses' Rule**, which combines the **reversal of conditioning**

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

and the total probability rule

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

If the events $A_1, \dots, A_k$ form a partition of the sample space,

$$P(A_i|B) = \frac{P(B|A_i)P(A_i)}{P(B|A_1)P(A_1) + \ldots + P(B|A_k)P(A_k)}$$

This is a more general form of Bayes' rule.

Enzyme Immunoassay Test for HIV

- $P(\text{T+}|\text{I+}) = 0.98$ (sensitivity - positive for infected)
- $P(\text{T-}|\text{I-}) = 0.995$ (specificity - negative for non-infected)
- $P(\text{I+}) = 1/300$ (prevalence in the US: estimated 1 million HIV infected)

What is the probability that the tested person is infected if the test was positive?

$$\begin{aligned} P(\text{I+}|\text{T+}) &= \frac{P(\text{T+}|\text{I+})P(\text{I+})}{P(\text{T+}|\text{I+})P(\text{I+}) + P(\text{T+}|\text{I-})P(\text{I-})} \\ &= \frac{0.98 \times 0.0033}{0.98 \times 0.0033 + 0.005 \times 0.9967} \\ &= 39.4\% \end{aligned}$$

This test is not confirmatory. Need to be confirmed by a second type of test