STAT22000 Autumn 2013 Lecture 10

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4.1 Randomness
4.2 Probability models

Randomness and Probability

Randomn $\neq$ Haphazard

- A phenomenon is **random** if individual outcomes are uncertain but there is nonetheless a regular distribution of outcomes in a large number of repetitions.
- A haphazard phenomenon may not have such long run distribution
  - e.g., Coin tossing is random. The long run proportion of heads is 0.5 (if using a fair coin).
  - e.g., Who will be the first 10 students show up in the classroom is haphazard.
- The **probability** of any outcome of a random phenomenon can be defined as the proportion of times the outcome would occur in a very long series of repetitions.

Probability Models

Probability models describe, mathematically, the outcome of random processes. They consist of two parts:

1. S = Sample Space: This is a set, or list, of all possible outcomes of a random process. An event is a subset of the sample space.
2. A probability for each possible event in the sample space S.

Examples

- Toss a coin and record the side facing up.
  $S = \{\text{Heads}, \text{Tails}\} = \{H, T\}$.
  Probability of heads = 0.5, Probability of tails = 0.5
- Toss a coin three times. Record the side facing up each time.
  $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
  Each outcome has probability 1/8.

More Examples of Probability Models

- Toss a coin 3 times and record the total number of heads.

$$S = \{0, 1, 2, 3\}$$

number of heads	0	1	2	3
probability	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

- Toss a coin repeatedly until a head occurs and record the total number of tosses. Then
  $S = \{H, TH, TTH, TTTH, \ldots\}$ or $S = \{1, 2, 3, 4, \ldots\}$

number of tosses	1	2	3	$\cdots$	n	$\cdots$
probability	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\cdots$	$\frac{1}{2^n}$	$\cdots$

All the examples listed above have discrete outcomes that we can list all the possible values. The sample space can also be continuous

- The life of a battery has a continuous sample space
  $S = [0, \infty)$

Events

An **event** A is a set of outcomes in the sample space.

Example 1: Toss a coin 3 times.
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

- Let A be the event that we get exactly 2 tails. Then $A = ?$
  $\{HTT, THT, TTH\}$
- Let B be the event that we get at least 1 head. Then $B = ?$
  $\{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

Example 2: Toss a coin repeatedly until a head occurs.
$S = \{1, 2, 3, \ldots\}$

- Let A be the event that no head occurs in the first 3 toss. Then $A = ?$ $\{4, 5, 6, \ldots\}$

Example 3: Measure the life of a battery. $S = [0, \infty)$

- Let B be the event that the battery is dead in 90 days. Then $B = ?$[0,90]

Set Notations

Suppose A and B are events in the sample space S.

- The **empty set** $\emptyset = \{\}$ is a subset of all sets.
- (A or B) $\equiv (A \cup B) \equiv$ (the **union** of A and B)
  A happens or B happens or both happen.
- (A and B) $\equiv (A \cap B) \equiv$ (the **intersection** of A and B)
  A and B both happens
- $(A \cap B = \emptyset) \equiv A$ and B are **disjoint**
  $\equiv A$ and B are **mutually exclusive**,
  A and B cannot happen at the same time
- $A^c \equiv$ the complement of A
  $\equiv$ all elements that are not in A.
  A does NOT happen

Example

Toss a coin twice. The sample space S is $\{HH, HT, TH, TT\}$. Let

A be the event that we get two heads,

B be the event that we get exactly one tail, and

C be the event that we get at least one head.

So,

$A = \{HH\}$ $\quad B = \{TH, HT\}$ $\quad C = \{HH, HT, TH\}$

- $A^c =$? $\{HT, TH, TT\}$
- A or $B =$? $\{HH, TH, HT\}$
- A and $B =$? $\{\} = \emptyset$
- $B^c =$? $\{HH, TT\}$
- A and $C =$? $\{HH\}$
- B or $C =$? $\{HH, TH, HT\}$

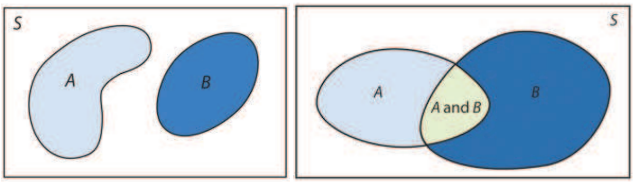
Probability Rules

The notation $P(A)$ means the probability of event A.

- **Rule 1**: All probabilities are between 0 and 1
- **Rule 2:** The probability of the whole sample space is 1:
$$P(S) = 1.$$
- **Rule 3 (Complement Rule):** The probability that A cannot occur is 1 minus the probability that A occurs
$$P(A^c) = 1 - P(A)$$
- **Rule 4 (Addition Rule):** If two events A and B are **disjoint** then
$$P(A \text{ or } B) = P(A) + P(B)$$
- **Rule 5 (General Addition Rule):** In general, for any two events A and B, the following rule always holds
$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Venn Diagram

Addition Rule $\qquad$ General Addition Rule



$P(A \text{ or } B) = P(A) + P(B)$

$P(A \text{ or } B)$
$= P(A) + P(B) - P(A \text{ and } B)$

Example — Complementation Rule

Question: What is the probability that there is at least one head in 3 tosses of a fair coin?

- Sample space
$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

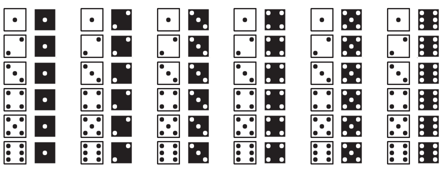
Event $A = \{\text{at least one head}\}$
$= \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$

- $A^c = \{TTT\}$
- $P(A^c) = 1/8$
- $P(A) = 1 - P(A^c) = 1 - 1/8 = 7/8$

Example — General Addition Rule

Rolling a pair of dice, what is the probability to get at least one ace?

State space:

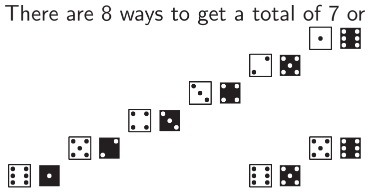


- Let A be the event {the white die is an ace}, $P(A) = 1/6$
- Let B be the event {the black die is an ace}, $P(B) = 1/6$
- Then the event of interest {at least one ace} is $A \cup B$.
- $A \cap B$={both are aces}={}, $P(A \cap B) = 1/36$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{6} + \frac{1}{6} - \frac{1}{36} = \frac{11}{36}$

Example — Addition Rule

Rolling a pair of dice, what is the probability to get a total of 7 or 11 spots?

- Event A: The total is 7 spots. $\mathbb{P}(A) =$ 6/36.
- Event B: The total is 11 spots. $\mathbb{P}(B) =$ 2/36.
- There are 8 ways to get a total of 7 or 11 spots.



So $\mathbb{P}(A \text{ or } B) = \frac{8}{36}$.

- In this example, the two events A and B are **disjoint** that you cannot get a total of 7 spots at the same time as getting a total of 11 spots. So
$$P(A \text{ or } B) = P(A) + P(B).$$