Counting

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When We Need to Count

A common probability model consists of a (possibly large) finite sample space, with each outcome having equal probability. Calculating the probabilities of events of interest, then, is a matter of **counting**:

- How many outcomes are in the event A?
- How many outcomes are outside the event A (i.e. in A^c)?
- How many outcomes are in S?

Of course, $|A| + |A^c| = |S|$, so we need to figure out two of these three quantities.

When does this come up?

- Rolls of a die
- Draws from a pack of cards

- Outcomes of lotteries
- All over the place!

Example

If I roll a die three times, what is the probability that I get the same number at least twice?

$$S = \left\{ \begin{matrix} (1,1,1) & (1,1,2) & \cdots & (1,1,6) \\ (1,2,1) & (1,2,2) & \cdots & (1,2,6) \\ \vdots & \vdots & \ddots & \vdots \\ (6,6,1) & (6,6,2) & \cdots & (6,6,6) \end{matrix} \right\}$$

- How many possible outcomes are there?
- How many of these have a repeated number?

We need some good strategies for counting.

Divide and Conquer

Our goal will be to split a "big question" that's hard to think about directly into "smaller problems" by considering different cases.

The cases should be

- **Exclusive** (i.e. non-overlapping)
- **Exhaustive** (i.e. nothing is forgotten)

The smaller problems can then be dealt with either directly or by dividing further.

Example

How many possible outcomes are there?

- How many outcomes are there that start with 1?
- How many outcomes are there that start with 2?
- How many outcomes are there that start with 3?

How many outcomes are there that start with 4?

How many outcomes are there that start with 5?

How many outcomes are there that start with 6?

Counting Tricks

Trick 1: Use Symmetry

For example, "How many outcomes start with a k?" has the same answer for $k = 1, 2, \ldots, 6$ in our example.

When using symmetry, be careful – it's easy to fool yourself!

Back to our example...

$$\boxed{\text{\# possible outcomes}} = 6 \times \boxed{\text{\# outcomes starting with 1}}$$
$$= 6 \times 6 \times \boxed{\text{\# outcomes starting with (1,1)}}$$
$$= 6 \times 6 \times 6$$
$$= 216$$

Trick 2: Recognize standard counting problems and use the known answer

We'll say more about this later.

Trick 3: Rephrase the Question

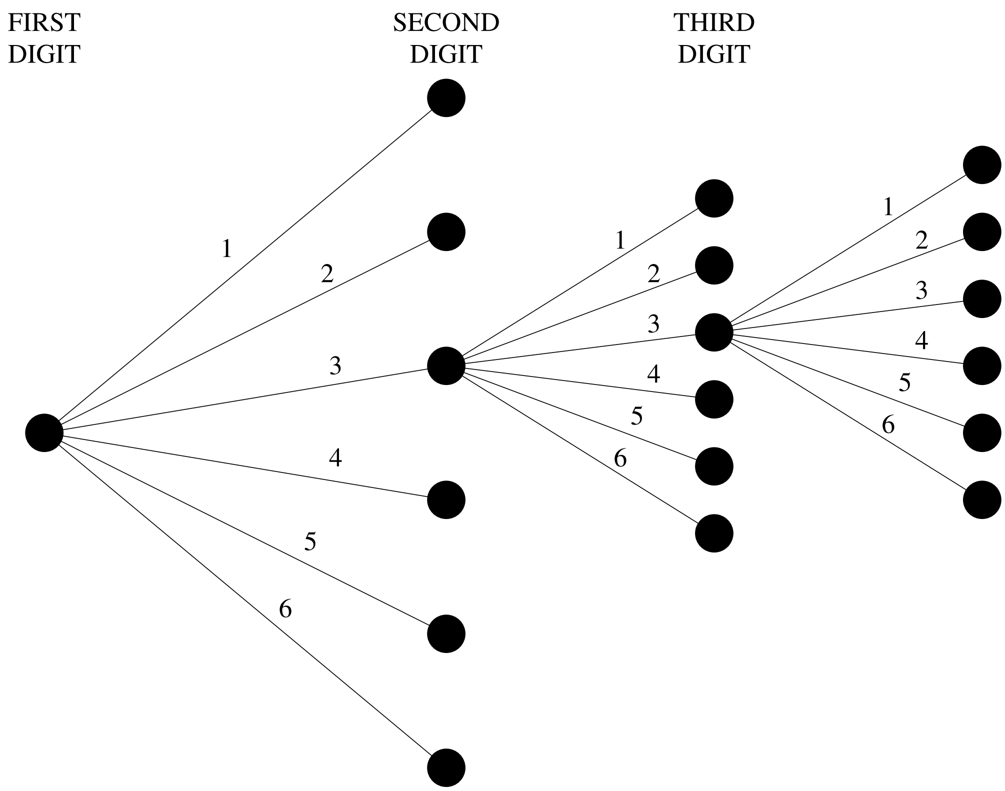
Back to our example...

$$\boxed{\text{\# outcomes with repeats}} = \underbrace{\boxed{\text{\# outcomes}}}_{\checkmark} - \underbrace{\boxed{\text{\# outcomes w/o repeats}}}_{\text{coming soon}}$$

Trick 4: Make an Exhaustive List by Brute Force

Trick 5: Check a Simpler Case

Trick 6: Draw a Tree Diagram


FIRST DIGIT
SECOND DIGIT
THIRD DIGIT
1
2
3
4
5
6
1
2
3
4
5
6
1
2
3
4
5
6

Trick 7: Try Different Ways of Breaking Up the Problem

This may require divorcing your thoughts from the actual process that's generating the outcomes.

Back to our example...

Decomposition 1:

- # outcomes with a repeat
  - # outcomes w/repeat starting with 1
    - 1 (1, 1, 1)
    - 10 1 repeated exactly twice
    - 5 only the initial 1 occurs
  - # outcomes w/repeat starting with 2
  - ⋮
  - # outcomes w/repeat starting with 6

By symmetry, the other cases are the

same. Therefore,

$$\boxed{\text{\# outcomes with a repeat}} = 6 \times (1 + 10 + 5)$$
$$= 96$$

Decomposition 2:

outcomes with a repeat

- 6 # outcomes with a triple
- # outcomes that repeat a digit exactly twice
  - 30 (a, a, b)
  - 30 (a, b, a)
  - 30 (b, a, a)

Therefore,

$$\boxed{\text{\# outcomes with a repeat}} = 6 + (30 + 30 + 30) = 96$$

What luck, we got the same answer!

Standard Counting Problems

Suppose you have n distinct objects and wish to select r of them. How many ways can this be done? (Note that, in order for the question to make sense, we require $0 \leq r \leq n$.)

We need to ask two questions:

- Are repeated selections OK?
- Does order matter? (In other words, are the possible outcomes *lists* or *sets* of elements?)

Example

- How many ways can we choose a President, VP, and Secretary from 20 employees?

  Are repeats allowed? Does order matter?

  The answer: $20 \times 19 \times 18 = 6,840$

- How many possible three-topping pizzas can be made with 10 ingredients?

  Are repeats allowed? Does order matter?

  The answer: $\binom{10}{3} = \frac{10\times9\times8}{3\times2} = 120$

Distinct Objects

Be aware that the n objects being selected from must be *distinct*.

Example

- How many ways can we arrange the letters A, B, C, D?

  Here, $n = r = 4$. The answer:

  $$_4P_4 = \frac{4!}{0!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24$$

- How many ways can we arrange the letters A, A, B, B?

  The ways are:

  $$\left\{\begin{matrix} AABB & ABBA & BABA \\ ABAB & BAAB & BBAA \end{matrix}\right\}$$

  so the answer is 6.

  What are the right distinct objects?

There are 4 slots, and we want to select 2 for the As. $\therefore n = 4,\ r = 2$. Repeats are not allowed and order does not matter.

Table of Standard Counting Problems

	Repeats	
Order	Not Allowed	Allowed
Matters	Permutation $_nP_r = \frac{n!}{(n-r)!}$	n^r
Doesn't Matter	Combination $_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$	**?**

Permutations

How many ways can we select an *ordered* arrangement of r objects from a set of n distinct objects? (Repeats are not allowed.)

- Any of the n objects could be first
- Any of the $n-1$ remaining objects could be second

  $\vdots$

- Any of the $n-r+1$ remaining objects could be r^{th}

$$\therefore \; {}_nP_r = (n)(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Back to our example...

How many outcomes have no repeats? In other words, what is ${}_6P_3$?

$$ {}_6P_3 = \frac{6!}{3!} = 6 \times 5 \times 4 = 120 $$

Thus, the total number of outcomes with a repeat is $216 - 120 = 96$

. . . the same answer again!

Combinations

How many ways can we choose r distinct objects from a set of n distinct objects, ignoring order?

$$ {}_nC_r = \binom{n}{r} = \frac{n!}{(n-r)!r!} $$

Note that ${}_nP_r = {}_nC_r \times {}_rP_r$. Thus, another way to decompose ${}_nP_r$ is to ask which of the ${}_nC_r$ collections of r objects was used, and then which of the ${}_rP_r$ possible arrangements of them was chosen.

Sampling with Replacement

What if we allow repeats? How many ways can we choose r objects *with possible repeats* from a set of n distinct objects?

- Any of the n objects could be first
- Any of the n objects could be second

  $\vdots$

- Any of the n objects could be r^{th}

$\therefore$ there are $\underbrace{n \times n \times \cdots \times n}_{r \text{ times}} = n^r$ possibilities.

Example: Poker Hands

A poker hand consists of 5 cards drawn from a standard deck of 52 cards (13 values in each of 4 suits).

For simplicity, we consider the order of the cards to be irrelevant. Thus, for example, $[A\heartsuit, 3\clubsuit, 5\spadesuit, 5\clubsuit, K\heartsuit]$ and $[A\heartsuit, 5\spadesuit, K\heartsuit, 3\clubsuit, 5\clubsuit]$ are the same hand. (In other words, a hand is a *set* of cards, not a *list* of cards.)

Note that if a poker hand falls into multiple "categories," we assign it to only the most valuable category. Thus, for example,

$$[3\heartsuit, 3\clubsuit, 5\spadesuit, 5\clubsuit, 3\diamondsuit]$$

contains two pairs, and three of a kind, but its most valuable designation is as a full house.

- How many distinct poker hands are there?
- How many straight-flush poker hands are there?
- How many four-of-a-kind poker hands are there?
- How many two-pair poker hands are there?

Example: A Final Word on Pizza

We have addressed how many pizzas with 3 distinct toppings can be made with 10 ingredients. (Recall the answer is $\binom{10}{3} = 120$.)

What if we allow repeated toppings? (This corresponds to the **?** cell in our table.)

Let's break the problem into cases:

- Case 1: All three toppings are distinct

  We've seen this: $\binom{10}{3} = 120$.

- Case 2: There are two orders of one topping and one of another topping

  In this case, we need to select two toppings out of the ten, and then choose one of the two to be duplicated:

$$\binom{10}{2} \times 2 = \frac{10 \times 9}{2} \times 2 = 90$$

- Case 3: There are three orders of a single topping

In this case, we just have to pick a topping out of the ten, so there are 10 possibilities.

Summing up, there are a total of $120 + 90 + 10 = 220$ possible pizzas.