Branching random walk in the presence of a hard wall

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Abstract

We consider the Branching Random Walk of height $n$ and show that the conditional expectation of a gaussian variable at a typical vertex, under positivity, is at least a factor of $\log n$ away from the expected maxima.

1 Introduction

Let us consider a $d$-ary tree of $n$ levels and call it $T_n$. We define a Branching random walk on $T_n$ and call it as $\{\phi^n_v : v \in T_n\}$.

We wish to find a bound on the order of the probability of a Branching Random Walk being positive at all vertices. We also wish to find the Expected value and Size of a typical vertex under the condition that it is positive everywhere. The behavior that we are considering is that of entropic repulsion for these Gaussian fields which is it’s behavior of drifting away when pressed against a hard wall so as to have enough room enough for local fluctuations, as is referred to in [8]. The phenomenon of entropic repulsion for gaussian free field has been studied in literature for some time now. The entropic repulsion for infinite Gaussian Free Field for dimension $\geq 3$ has been studied in [2]. As a continuation to this, the finite field with zero boundary conditions for dimension $\geq 3$ was studied in [4]. The two papers computed the order of the probability of positivity in the two different cases, and though the typical behavior of a vertex was similar, this order was not so, when positivity for the entire box was considered. But on removing positivity condition for a layer near the box, the order was same as in [2]. It was also shown in [4] that the probability of positivity in case of GFF in a box of dimension 2 decays exponentially, and this is really a boundary phenomenon. So in order to look into the long range correlations, and local fluctuations, the boundary effect has to removed. This approach has been taken in [1] to look into the behavior of a typical vertex when pressed against this hard wall. Other works in this regard have been done in case of the Gaussian membrane model in [7], in the critical dimension of 4. An extension of the work in case of GFF for dimension $\geq 3$ has been covered in [3].

We are interested in $P(\phi^n_v \geq 0 \ \forall v \in T^n)$ as well as $E(\phi^n_u | \phi^n_v \geq 0 \ \forall v \in T_n)$ and $\text{Var}(\phi^n_u | \phi^n_v \geq 0 \ \forall v \in T_n)$. Let us first define the expected maximum of the Branching random walk to be $m_n = E(\max_{v \in T^n} \phi_v)$. In [1] it has been shown that the conditional expectation under positivity is roughly close to the expected maximum for the discrete GFF in 2 dimensions. Here we show that for a branching random walk the conditional expectation is at least a constant times $\log n$ less than the expected maximum. The main result of this paper is:

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Theorem 1.1. There exists positive numbers $a, b$, $a < b$ such that for all $v \in T_n$,

$$m_n - b \log n + O(1) \leq \mathbb{E}(\phi_n^v \mid \phi_n^v \geq 0 \forall v \in T_n) \leq m_n - a \log n + O(1).$$

The approach that we take for proving this is that we raise the average value of the Gaussian process and then multiply a compensation probability to that. We optimize this average value so as to maximize the probability of positivity. The value at which this probability is maximized should ideally be the required conditional expectation.

In order to prove this in details, we invoke a new model called the Switching Sign Branching Random Walk, which is similar in structure to the original Branching random walk. Section 2 contains the definition of this model followed by a comparison of positivity for the Branching Random Walk with this model using Slepian’s Lemma. A left tail computation for the maxima of this model gives us the order of positivity of the branching Random Walk which is the concluding result of Section 2. Section 3 contains the main proof of the paper. The Upper bound follows from Section 2, while for the lower bound we further have to invoke the Bayes rule and tail estimates to arrive at our result.

Throughout the paper we will use $d_T$ to denote the tree distance. We term the Probability of the event $\{\phi_n^v \geq 0 \forall v \in T_n\}$ as $\Lambda^+_n$. First let us consider the sum of all the Gaussian variables at the level $n$ and term it $S_n$. In mathematical terms $S_n = \sum_{v: v \in T_n} \phi_n^v$, where the sum contains $d^n$ terms.

2 Switching Sign Branching Random Walk

At this juncture we start defining this new Gaussian process on the tree, which we call the Switching Sign Branching Random Walk. This was used to approximate the branching random walk in [5] in case of a 4-ary tree. We have generalized the process for a general d-ary tree. This process is different from the normal branching random walk in the sense that instead of the $d$-edges coming out of it being associated to independent normal random variables, they are associated to linear combinations of $d-1$ independent Gaussians, such that the covariance between any two of them is the same, and all of them add up to zero. The existence of this is guaranteed by the following Lemma.

Lemma 2.1. There exists $A \in \mathbb{R}^{(d-1) \times (d-1)}$ such that for $X \sim N(0, \sigma^2 I_{(d-1) \times (d-1)})$, the covariance matrix of $AX$ has all it’s diagonal entries to be $\sigma^2$ and all it’s off-diagonal entries to be equal (say $b$). Further $\text{Var}(1^T AX) = \sigma^2$ and $\text{Cov}(-1^T AX, (AX)_i) = b$ for all $i \in \{1, 2, \ldots, d-1\}$.

Proof. We know that the covariance matrix for $AX$ is $AA^T$. Further from the condition that $\text{Var}(1^T AX) = \sigma^2$ we get that $b = -\frac{\sigma^2}{d-1}$. So in order for $A$ to exist we must have

$$AA^T = \sigma^2 \begin{bmatrix}
1 & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\
-\frac{1}{d-1} & 1 & -\frac{1}{d-1} & \cdots & -\frac{1}{d-1} \\
-\frac{1}{d-1} & -\frac{1}{d-1} & 1 & \cdots & -\frac{1}{d-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\frac{1}{d-1} & -\frac{1}{d-1} & -\frac{1}{d-1} & \cdots & 1
\end{bmatrix}_{(d-1) \times (d-1)}.$$

Since the matrix on the right hand side is a symmetric matrix with positive eigenvalues, so by Cholesky decomposition we obtain the existence of such an $A$. 

\[\square\]
Now in the actually construction, unlike the BRW, we use a different value for $\sigma^2$ for each level $l$ such that $1 \leq l \leq n$. Here level 1 denotes the edge connecting the root to its children and level $n$ denotes the edges joining the leaf nodes to their parents. Let us denote this switching signs branching random walk on the tree $T_n$ as $\{\xi^n_v : v \in T_n\}$. For $v \in T_n$ let us denote the Gaussian variable that is added on level $l$, on the path connecting $v$ to the root, by $\phi^{n,l}_v$. Then we assign $\text{Var}(\phi^{n,l}_v) = 1 - d^{-n} (n - l + 1)$. The switching sign branching random walk will consist of two parts, the first coming from the contribution at different levels in the tree which we call $\tilde{\phi}^n_v \overset{\text{def}}{=} \sum_{l=1}^n \phi^{n,l}_v$.

Finally we define the switching signed branching random walk as

$$\xi^n_v = \tilde{\phi}^n_v + X$$

where $X$ is a gaussian variable with mean zero and variance $\frac{1-d^{-n}}{d-1}$.

The covariance structure for this new model closely resembles that of the Branching random Walk. The following lemma deals with this comparison:

**Lemma 2.2.** The Gaussian fields $\{\xi^n_v : v \in T_n\}$ and $\{\phi^n_v : v \in T_n\}$ are identically distributed.

**Proof.** First we show that the variances are identical for the two processes. To this end,

$$\text{Var}(\xi^n_v) = 1 - d^{-1} + 1 - d^{-2} + \cdots + 1 - d^{-n} + \frac{1 - d^{-n}}{d-1} = n.$$

Next in case of the covariances suppose we consider $u, v \in T_n$, such that they are separated until level $k$ i.e $\text{Cov}(\phi^n_u, \phi^n_v) = n - k$. Then we have

$$\text{Cov}(\tilde{\phi}^n_u, \tilde{\phi}^n_v) = -\frac{1 - d^{-k}}{d-1} + \sum_{l=k+1}^n (1 - d^{-l}) = n - k - \frac{1 - d^{-n}}{d-1}.$$

So, the covariance structures for the fields $\xi$ and $\phi$ match, and hence they are identically distributed.

A simple corollary of Lemma 2.2 is the following, based on the fact that the two processes have identical distributions.

**Corollary 2.3.** We have the following equality:

$$\mathbb{P}(\phi^n_v \geq 0 \ \forall v \in T_n) = \mathbb{P}(\max_{v \in T_n} \tilde{\phi}^n_v \leq X)$$

(2)

From the (2) we understand that the probability of positivity for the Branching Random Walk can be computed using bounds on the Left tail of the maxima of $\tilde{\phi}^n$, a part of the switching sign Branching Random Walk, as the Left tail is heavily concentrated around the maxima. This motivates the following computations on the Left tail of the maxima.

**Lemma 2.4.** Let us call $c$ to be the constant such that $|m_n - c\lambda - m_n - \lambda| \rightarrow 0$ as $n \rightarrow \infty$, where $\lambda$ is of lower order than $n$. Then there exists independent constants $C', C'', K', K$ such that for sufficiently large $n$ we have

$$K'\left(\frac{\bar{p}}{K\lambda}\right)^{d\lambda} \leq \mathbb{P}(\max_{v \in T_n} \tilde{\phi}^n_v \leq m_n - \lambda) \leq C' \exp(-C''d^{\lambda}).$$

(3)

where $\bar{p}$ is a such that $0 < \bar{p} < 1$ and is independent of $n$. 

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Proof. We work with $\mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \leq m_{n-c\lambda})$ as due to our definition of $c$, for sufficiently large $n$ this probability is infinitesimally close to $\mathbb{P}(\max_{v \in T^n} \phi_v \leq m_n - \lambda)$. This comes from the fact that, from [6], $\{\max_{v \in T^n} \tilde{\phi}_v - m_n\}$ converges in distribution, and so by an application of Slutsky’s theorem $\mathbb{P}(\max_{v \in T^n} \phi_v \leq m_{n-c\lambda})$ and $\mathbb{P}(\max_{v \in T^n} \phi_v \leq m_n - \lambda)$ converge to the same value. We know that the switching sign branching random walk is a log-correlated gaussian field which lets us find the expected maxima, apart from guaranteeing the convergence in distribution of the recentered maxima. We first consider the tree only up to the level $c\lambda$ and consider the cumulative sum of the Gaussian variables at these vertices till the level $c\lambda$. Let us rename all these Gaussian variables at level $c\lambda$ of this new tree to be $A_1, A_2, \ldots, A_{d\lambda}$. We know that the switching sign branching random walk model guarantees $\sum_{i=1}^{d\lambda} A_i = 0$. Let us consider the subtrees rooted at the vertex which has values $A_i$ and call its maxima to be $M_i$. These are trees of height $n-c\lambda$ and hence we have $\mathbb{E}M_i = m_{n-c\lambda} \quad \forall i$ and $M := \max_{v \in T^n} \phi_v = \max_{i=1}^{d\lambda}(M_i + A_i)$. We want to obtain bounds for the probability $\mathbb{P}(\max_{v \in T^n} \phi_v \leq m_{n-c\lambda})$. We condition on the values of $A_1, A_2, \ldots, A_{d\lambda}$ which in turn breaks down the required probability in a product form since the maxima for the $d\lambda$ subtrees are independent and have identical distributions. We consider two different cases:

1) When $A_i^- \leq 1$ for at least $d\lambda/2$ many $i$.

2) When 1) doesn’t happen and so then $\sum_{i=1}^{d\lambda} A_i^- \geq d\lambda/2$.

Even for the first case we break it down into two parts according to when $\sum_{i=1}^{d\lambda} A_i^- \geq d\lambda/2$ and when it is not. Now we have

\[
\mathbb{P}(\max_{v \in T^n} \phi_v \leq m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d\lambda})
= \mathbb{P}(\max_{i=1}^{d\lambda}(M_i + A_i) \leq m_{n-c\lambda} \mid A_1, A_2, \ldots, A_{d\lambda})
= \Pi_{i=1}^{d\lambda} \mathbb{P}(M_i + A_i \leq m_{n-c\lambda} \mid A_i)
= \Pi_{i=1}^{d\lambda} \mathbb{P}(M_i \leq m_{n-c\lambda} - A_i \mid A_i)
\leq \Pi_{i=1}^{d\lambda} \mathbb{P}(M_i \leq m_{n-c\lambda} - A_i \mid A_i) I_{\{A_i > 0\}}
\leq \exp(-c^* \sum_{i=1}^{d\lambda} A_i^+) = \tilde{C}' \exp(-c^* \sum_{i=1}^{d\lambda} A_i^-)
\]

In the final two steps we first make use of [6] Lemma 2.1, followed by the fact that $\sum_i A_i = 0$. For the cases where $A_i < 0$ we replace the terms by 1. When 2) holds then clearly this is bounded by $\exp(-C^{**}d\lambda)$ and in the other case also

\[
\mathbb{P}(M_i \leq m_{n-c\lambda} - A_i \mid A_i) \leq \mathbb{P}(M_i \leq m_{n-c\lambda} + 1)
\]

for those $i$ for which $A_i^- \leq 1$. From the convergence in distribution for recentered maxima, we can find $p$, independent of $n$, where $0 < p < 1$ such that $\mathbb{P}(M_i \leq m_{n-c\lambda} + 1) < p$ for all sufficiently large $n$ and so the probability is bounded by $\exp(-\tilde{c}d\lambda)$. Now from this $\tilde{c}$ and $c^{**}$ we select one unified $C', C''$ so that

\[
\mathbb{P}(\max_{v \in T^n} \tilde{\phi}_v \leq m_{n-c\lambda}) \leq C' \exp(-C''d\lambda).
\]
In (2), we condition on the value of $X$ of a Gaussian, the first part is bounded by

$$\Pr(\max_{v \in T^n} \hat{\phi}_v \leq m_{n-c\lambda}) = \int_{\mathbb{R}^{d\lambda}} \prod_{i=1}^{2c\lambda} \Pr(M_i \leq m_{n-c\lambda} - A_i) dA_i \geq (\bar{p})^{d\lambda} \int_{[-1,1]}^{d\lambda} \prod_{i=1}^{d\lambda} dA_i$$

where $\bar{p}$ is chosen to be a lower bound on $\Pr(M_i \leq m_{n-c\lambda} - 1)$ for all sufficiently large $n$, which exists due to convergence in distribution of recentered maxima of a log correlated Gaussian field. Now $\{A_1, A_2, \ldots, A_{d\lambda}\}$ are obtained by linear combinations of $d\lambda - 1$ independent standard normal random variables, each being obtained from $c\lambda$ many of them, and a way to make all $A_i$'s in the range $[-1, 1]$ is to make the independent standard normals in the range $[-1/c\lambda \sqrt{d}, 1/c\lambda \sqrt{d}]$. So,

$$\Pr(\max_{v \in T^n} \hat{\phi}_v \leq m_n - \lambda) \geq K'(\frac{\bar{p}}{K\lambda})^{d\lambda}$$

We now look back into our question of the Branching Random Walk being positive at all vertices. By property of log-correlated fields, the maxima is heavily concentrated around the expected maxima. In a neighborhood around the maxima, we further try to maximize the probability of the maxima being there. This point where this occurs will also roughly be the typical value of a vertex. This motivates the following theorem which is the main result of this section:

**Proposition 2.5.** There exists $\lambda'$ such that $d\lambda'$ is of order $n$ such that for $n$ sufficiently large we have, for $K_1, K_2 > 0$ independent of $n$,

$$K_1 e^{-\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda')^2 - d\lambda' \log \lambda' - \log K} \leq \Pr(\phi_v^n \geq 0 \ \forall v \in T_n) \leq K_2 e^{-\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda')^2 - \frac{m_n - \lambda'}{\sigma_{d,n} \log d}}. \quad (4)$$

**Proof. Upper bound:** From (2) we have an upper-bound on the probability of positivity based on the Switching Signs Branching Random walk. We optimize this bound by first raising the mean to a level and look at the compensation we have to apply correspondingly. We optimize over these two to obtain our bound. We apply a similar strategy for obtaining the lower bound as well. Let us recall (2) at this juncture along with $X$, and let us call the variance of $X$ to be $\sigma_{d,n}^2 = \frac{1-d^{-n}}{d-1}$. In (2), we condition on the value of $X$ to obtain the following:

$$\Pr(\Lambda_n^+) = \frac{1}{\sigma_{d,n} \sqrt{2\pi}} \int_{-\infty}^{\infty} \Pr(\max_{v \in T_n} \hat{\phi}_v^n \leq x) \exp(-x^2/2\sigma_{d,n}^2) dx$$

Now, since the left tail of the maxima of a log-correlated Gaussian field, is heavily concentrated. So we may as well replace $x$ by $m_n - \lambda$, and then integrate over $\lambda$. We split the integral into three parts, first with $\{-\infty < \lambda \leq 0\}$, second with $\{\frac{\lambda}{2} \log d n \leq \lambda < \infty\}$ and the rest. From tail estimates of a gaussian, the first part is bounded by $O(\exp(-\frac{1}{2\sigma_{d,n}^2}(m_n - \lambda'))^2)$). From (3), we know that the second part is bounded by $C' \exp(-C''n^3)$. The rest part has an upper bound:

$$\frac{C'}{\sqrt{2\pi}} \int_{0}^{\frac{1}{2} \log d n} \exp(-C''d\lambda) \exp(-(m_n - \lambda)^2/2) d\lambda. \quad (5)$$

We maximize the integrand in (5), over the range of the integral, to obtain an optimal $\lambda'$, which is of order $\log n$. It satisfies the equation

$$m_{n-1} - \lambda' = \sigma_{d,n}^2 C' d e^{c\lambda} \log d.$$
Plugging in we obtain an upper bound as in [4].

**Lower bound:** Again recalling (3) we obtain that

\[
\mathbb{P}(\Lambda_+^n) \geq \frac{K'}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} e^{-(m_n - \lambda)^2/2\sigma_{d,n}^2} d\lambda.
\]

The integrand here is fact a decreasing function of \( \lambda \) in the range \( \lambda \in [\lambda', \lambda' + 1] \), where \( \lambda' \) is from the first part of the proof. This gives a lower bound of

\[
\frac{K'}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} e^{-(m_n - \lambda)^2/2\sigma_{d,n}^2} d\lambda.
\]

So, we obtain the required lower bound in [4].

\[\square\]

### 3 Proof of Theorem 1.1

**Proof.** We want to compute \( \mathbb{E} \left( \frac{S}{\hat{v}_n} \mid \Lambda_+^n \right) \). Due to Lemma 2.2 this is equivalent to computing

\[
\mathbb{E} \left( \sum_{v=1}^{d_n} \hat{S}_v \mid \hat{\epsilon}_v^n \geq 0 \ \forall \ v \in T_n \right) = \mathbb{E} \left( X \mid \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right).
\]

**Upper Bound:** We first split the expectation into two parts, one concerning the contribution of the right tail in the integral and the rest. We aim to show that the contribution of the right tail is negligible, thereby implying that the main contribution is from the rest, which gives an upper bound on the expectation. The tail here is motivated by the maximizer in Proposition 2.5

\[
\mathbb{E} \left( X \mid \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right) = \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq x \right)}{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right)} dx
\]

\[
= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{-\infty}^{m_n-b\log n} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq x \right)}{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right)} dx
\]

\[
+ \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n-b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq x \right)}{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right)} dx
\]

Let us call the first term as \( J_1 \) and the next one as \( J_2 \).

\[
J_2 \leq \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \int_{m_n-b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{1}{\mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right)} dx
\]

\[
= \frac{1}{\sqrt{2\pi}\sigma_{d,n}} \mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right) \int_{m_n-b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right) \int_{m_n-b\log n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} dx
\]

\[
= \frac{\sigma_{d,n} e^{-(m_n-b\log n)^2/2\sigma_{d,n}^2}}{\sqrt{2\pi} \mathbb{P}\left( \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right)}
\]

\[\square\]
\[ J_1 \leq \frac{m_n - b \log n}{\sqrt{2\pi} \sigma_{d,n}} \int_{-\infty}^{m_n - b \log n} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx \]
\[ \leq \frac{m_n - b \log n}{\sqrt{2\pi} \sigma_{d,n}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx \]
\[ = m_n - b \log n \]

From (4) it is clear that on choosing \( b \) such that \( b \log n \leq \lambda' \) then the upper bound on the conditional expectation is \( m_n - b \log n \).

**Lower Bound:** We apply a similar technique as in case of the upper bound, the only difference being that we look at the left tail instead, motivated by the left tail of the maxima of the Gaussian process.

\[
\mathbb{E}\left( X \mid \max_{v \in T_n} \tilde{\phi}_v^n \leq X \right) = \frac{1}{\sqrt{2\pi} \sigma_{d,n}} \int_{-\infty}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx
\]
\[ = \frac{1}{\sqrt{2\pi} \sigma_{d,n}} \int_{m_n - \frac{2}{c} \log d \, n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx
\]
\[ + \frac{1}{\sqrt{2\pi} \sigma_{d,n}} \int_{m_n - \frac{2}{c} \log d \, n}^{\infty} x e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx
\]

Let us call the first term as \( I_1 \) and the second as \( I_2 \).

When \( x \in (-\infty, m_n - \frac{2}{c} \log d \, n] \) then \( \mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x) \leq C' \exp(-C'' n^3) \) following (3). Also we have a lower bound on the probability of positivity, which gives the following bounds on \( I_1 \) and \( I_2 \).

\[ |I_1| \leq e^{\frac{1}{2\sigma_{d,n}}(m_n - \lambda')^2 + d e^{\lambda'}(\log \lambda' - \log \rho/K) - C'' n^3} \int_{-\infty}^{\infty} \frac{x}{e^{-x^2/2\sigma_{d,n}^2}} \, dx
\]

where we ignore the constants. This shows that this term is negligible. Further,

\[ I_2 \geq \left( m_n - \frac{3}{c} \log d \, n \right) \frac{1}{\sqrt{2\pi} \sigma_{d,n}} \int_{m_n - \frac{2}{c} \log d \, n}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx
\]
\[ = \left( m_n - \frac{3}{c} \log d \, n \right) \frac{1}{\sqrt{2\pi} \sigma_{d,n}} \int_{-\infty}^{\infty} e^{-x^2/2\sigma_{d,n}^2} \frac{\mathbb{P}(\max_{v \in T_n} \tilde{\phi}_v^n \leq x)}{\mathbb{P}(\max_{v \in T_n} \phi_v^n \leq X)} \, dx - o(1)
\]
\[ = m_n - \frac{3}{c} \log d \, n
\]

\[ \square \]

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References


