The Leaky Integrate-and-Fire neuron
Single neuron models

- Hodgkin-Huxley (HH) model
  - Good at capturing quantitatively diversity of dynamical features of real neurons
  - Highly non-linear, large number of variables $\Rightarrow$ hard to analyze mathematically
  - Simulations of networks of HH neuron computationally expensive

- Leaky integrate-and-fire (LIF) neuron
  - Too simple to reproduce diversity of dynamical features of real neurons
  - Can be analyzed mathematically in great detail (at both neuron and network levels)
  - Possible to simulate very large networks of such neurons

- Models in between HH and LIF: try to capture best of both worlds
  - Mathematically tractable
  - Dynamical behaviors of HH-type models (and real neurons)
Leaky integrate-and-fire (LIF) neuron

- Subthreshold dynamics ($V < V_T$): keep only capacitive and leak currents (Lapicque 1907)
  \[
  C \frac{dV}{dt} = -g_L(V - V_L) + I_{syn}(t) \\
  \tau_m \frac{dV}{dt} = -V + \tilde{I}_{syn}(t)
  \]
  - Spike emitted when $V = V_t$;
  - Then voltage reset to $V = V_r$;
  - (Optional) absolute refractory period of duration $\tau_{rp}$
  - $V_t$, $V_r$ and $\tau_{rp}$ replace Na and K currents in HH model
Constant input - f-I curve

- When $I$ is constant, firing is periodic;
- To compute inter-spike interval $T$, solve
  \[ \tau_m \frac{dV}{dt} = -V + I \]
  with initial condition $V(0) = V_r$; $T$ is given by $V(T) = V_T$
- Yields
  \[ T = \tau_m \log \left( \frac{I - V_r}{I - V_T} \right) \]
- With non zero ARP,
  \[ T = \tau_{rp} + \tau_m \log \left( \frac{I - V_r}{I - V_T} \right) \]
- Mean firing frequency given by
  \[ f = \frac{1}{T} = \frac{1}{\tau_{rp} + \tau_m \log \left( \frac{I - V_r}{I - V_T} \right)} \]
Large fluctuations in inputs *in vivo*

Haider et al 2013
Origin of large fluctuations

• Neurons in vivo are always active (spontaneous activity, $\nu \sim 0.1-10$/s)
• Neurons have a large number of synapses on their dendritic trees ($C \sim 10,000$)
• Hence, neurons are constantly bombarded by synaptic inputs ($C\nu \sim 10,000$)
Irregularity of spike trains in vivo

## Standard ISI distributions

<table>
<thead>
<tr>
<th>pdf of ISIs $P(T) =$</th>
<th>Poisson</th>
<th>Gamma $\frac{(k\nu)^k}{\Gamma(k)} T^{k-1} \exp(-k\nu T)$</th>
<th>Inverse Gaussian $\sqrt{\frac{k}{2\pi T^3 \nu}} \exp\left(-\frac{k(\nu T - 1)}{2T \nu}\right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1/\langle T \rangle =$</td>
<td>$\nu$</td>
<td>$\nu$</td>
<td>$\nu$</td>
</tr>
<tr>
<td>$\frac{SD(T)}{\langle T \rangle}$</td>
<td>$1$</td>
<td>$\frac{1}{\sqrt{k}}$</td>
<td>$\frac{1}{\sqrt{k}}$</td>
</tr>
</tbody>
</table>

![Graphs showing pdf curves for different lambda values](image1)

![Graphs showing pdf curves for different k values](image2)
The Poisson process

- Specific type of a renewal process (ISIs are independent of previous history)
- Each ISI drawn randomly from $P(T) = \nu \exp(-\nu T)$ where $\nu$ is the rate
- In any interval $[t, t + T]$ the number of spikes is given by a Poisson distribution,
  \[ P(k) = (\nu T)^k \exp(-\nu T)/k! \]
LIF model with Poisson inputs

- Consider a LIF neuron with Poisson inputs
  \[ \tau \dot{V} = -V + I(t) \]

- \( I(t) \) Poisson process with rate \( \nu_{in} \)
  \[ I(t) = J \tau \sum_k \delta(t - t_k) \]

- At each spike, \( V(t_k^+) = V(t_k^-) + J \)

- Between spikes, voltage decays exponentially

- Mean(\( V \)) = \( J \tau \nu_{in} \)

- Var(\( V \)) = \( J^2 \tau \nu_{in} / 2 \)
Master equation

- \( P(V, t) \) is described a master equation
  \[
  \frac{\partial P(V, t)}{\partial t} = \frac{1}{\tau} \frac{\partial}{\partial V} (VP(V, t)) + \nu_{in} [P(V - J, t) - P(V, t)]
  \]

- Can be rewritten as a continuity equation:
  \[
  \frac{\partial P}{\partial t} = -\frac{\partial S}{\partial V}
  \]
  where \( S \) is probability flux
  \[
  S(V, t) = \nu_{in} \int_{V-J}^{V} P(w, t)dw - \frac{VP}{\tau}
  \]

- Boundary conditions \( \Rightarrow \) links \( P \) and instantaneous firing probability \( \nu_{out} \)
  - At threshold \( V_T \):
    \[
    \nu_{out} = S(V_T, t)
    \]
  - At reset potential \( V_R \): what comes out at \( V_T \) must come back at \( V_R \),
    \[
    \nu_{out} = S(V_R^+, t) - S(V_R^-, t)
    \]
Diffusion approximation

- Diffusion approximation:

\[ I(t) = J \tau \sum_k \delta(t - t_k) \approx \mu + \sigma \sqrt{\tau} \eta(t) \]

where \( \mu = J \nu_{in} \tau \), \( \sigma^2 = J^2 \nu_{in} \tau \) and \( \eta(t) \) is a white noise (\( \langle \eta(t) \rangle = 0 \), \( \langle \eta(t) \eta(t') \rangle = \delta(t - t') \)).

- Voltage = Ornstein-Uhlenbeck process, obeys a Langevin equation

\[
\tau \frac{dV}{dt} = -V + \mu + \sigma \sqrt{\tau} \eta(t)
\]

(physics notation)

\[
dV = \frac{\mu - V}{\tau} dt + \frac{\sigma}{\sqrt{\tau}} dB
\]

(math notation) where \( B \) is a Wiener process (white noise)

- Discrete version:

\[
V(t + dt) = V(t) + \frac{dt}{\tau} (\mu - V(t)) + \sqrt{\frac{dt}{\tau}} \sigma Z
\]

where \( Z \) is a random Gaussian variable of zero mean and unit variance
Fokker-Planck equation and boundary conditions

- Consider a LIF neuron with deterministic + white noise inputs,

\[ \tau \dot{V} = -V + \mu(t) + \sigma(t) \eta(t) \]

- \( P(V, t) \) is described by Fokker-Planck equation

\[ \tau_m \frac{\partial P(V, t)}{\partial t} = \frac{\sigma^2(t)}{2} \frac{\partial^2 P(V, t)}{\partial V^2} + \frac{\partial}{\partial V} \left[ (V - \mu(t))P(V, t) \right] \]

- Boundary conditions \( \Rightarrow \) links \( P \) and instantaneous firing probability \( \nu \)

  - At threshold \( V_t \): absorbing b.c. + probability flux at \( V_t = \) firing probability \( \nu(t) \):

\[ P(V_t, t) = 0, \quad \frac{\partial P}{\partial V}(V_t, t) = -\frac{2\nu(t)\tau_m}{\sigma^2(t)} \]

  - At reset potential \( V_r \): what comes out at \( V_t \) must come back at \( V_r \)

\[ P(V_r^-, t) = P(V_r^+, t), \quad \frac{\partial P}{\partial V}(V_r^-, t) - \frac{\partial P}{\partial V}(V_r^+, t) = -\frac{2\nu(t)\tau_m}{\sigma^2(t)} \]
LIF model: equations for f-I curve and CV

For constant $\mu = \mu_0$ and $\sigma = \sigma_0$:

$$ P_0(V) = \frac{2\nu_0 \tau_m}{\sigma} \exp \left( -\frac{(V - \mu_0)^2}{\sigma^2} \right) \int_{\frac{V - \mu_0}{\sigma}}^{\frac{V_t - \mu_0}{\sigma}} \exp(u^2) \Theta(u - V_r) du $$

$$ \nu_0 = \frac{1}{\tau_m \sqrt{\pi} \int_{\frac{V_t - \mu_0}{\sigma}}^{\frac{V_r - \mu_0}{\sigma}} \exp(u^2)[1 + \text{erf}(u)]} $$

$$ CV^2 = 2\pi \nu_0^2 \int_{\frac{V_t - \mu_0}{\sigma}}^{\frac{V_r - \mu_0}{\sigma}} e^{x^2} dx \int_{-\infty}^{x} e^{y^2} (1 + \text{erf}y)^2 dy $$
LIF model: f-I curve, CV-I curve, P(V)

Sub-threshold

Supra-threshold
f-I curves in presence of noise

Rauch et al 2003
f-I curves in presence of noise

Rauch et al 2003
Given an arbitrary time-dependent input \((\mu(t), \sigma(t))\) what is the instantaneous firing rate \(\nu(t)\)?
Computing the linear firing rate response

- Strategy:
  
  - start with small time-dependent perturbations around means,
    
    \[ \mu(t) = \mu_0 + \epsilon \mu_1(t), \quad \sigma(t) = \sigma_0 + \epsilon \sigma_1(t) \]

  - linearize FP equation and obtain the linear response of \( P = P_0 + \epsilon P_1(t) \) and \( \nu = \nu_0 + \epsilon \nu_1(t) \) (solution of inhomogeneous 2nd order ODE).

  \[
  \nu_1(t) = \int_0^t R_\mu(t-t')\mu_1(t') + R_\sigma(t-t')\sigma_1(t')dt' \\
  \tilde{\nu}_1(\omega) = R_\mu(\omega)\tilde{\mu}_1(\omega) + R_\sigma(\sigma)\tilde{\sigma}_1(\omega)
  \]

  - \( R_\mu \) and \( R_\sigma \) can be computed explicitly in terms of confluent hypergeometric functions.

  - go to higher orders in \( \epsilon \)...
LIF model: linear rate response

- Response of the instantaneous firing rate to oscillatory inputs at frequency $f$ + noise

$$I_{syn}(t) = I_0 + I_1 \cos(2\pi ft) + I_{noise}(t)$$

$$\Rightarrow \nu(t) = \nu_0 + \nu_1(f) \cos(2\pi ft - \phi(f))$$

- Resonances at $f = n\nu_0$
  for high rates and low noise;

- Attenuation at high $f$

  Gain $\sim \begin{cases} 
  \frac{1}{\sqrt{f}} & \text{(white noise)} \\
  \sqrt{\frac{\tau_s}{\tau_m}} & \text{(colored noise)}
  \end{cases}$

- Phase lag at high $f$

  Lag $\sim \begin{cases} 
  \frac{\pi}{4} & \text{(white noise)} \\
  0 & \text{(colored noise)}
  \end{cases}$
Bibliography

• Burkitt review papers in Biol.Cybern. (2006)

More on stochastic processes:

• Gardiner, Stochastic Methods: A Handbook for the Natural and Social Sciences

• Risken, The Fokker-Planck Equation: Methods of Solution and Applications

• van Kampen, Stochastic Processes in Physics and Chemistry