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The interpolation of options

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Abstract. Conservative delta hedging permits traders to pass from nonparametric bounds on interest rates and volatilities to trading strategies. The uncertainty reflected in the bounds, however, provides relatively high starting prices for these strategies. We here show how the effect of uncertainty to a substantial extent can be offset by interpolation, i.e., the hedging in auxiliary market traded securities. As before, the technology does not involve the specification of models for the term structure of volatilities and interest rates, and should therefore be particularly appropriate from the point of view of risk management.

Key words: Conservative delta hedging, incompleteness, statistical uncertainty, value at risk

JEL Classification: C10, D52, D81, G13

Mathematics Subject Classification (1991): 60H30, 60G44, 60F05, 62F03, 62P05

1 Introduction

Conservative delta hedging (Mykland 2000, 2001) provides a device for passing from nonparametric bounds on cumulative interest rates and volatilities to hedging strategies for options. The resulting strategies also provide intervals for the value of options. A main problem with the approach is that such intervals can often be uncomfortably wide. The purpose of this paper is to show how one can reduce this problem by involving auxiliary securities, in particular, market traded options.

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The setting is the securities price model

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{1.1}$$

where W is a Brownian motion and where μ_t, σ_t and the risk free short rate r_t can be stochastic and random.

The problem is that the probability distribution P is unknown. We here take P to mean the actual probability distribution, as opposed to equivalent martingale measures, denoted P^*. Hence, this is a double problem: both P and P^* are unknown.

To deal with this problem, conservative delta hedging starts with prediction intervals of the form

$$R^+ \geq \int_0^T r_u du \geq R^- \text{ and } \Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-, \tag{1.2}$$

and convert this into bid-ask intervals on the form $[A, B]$, where

$$A = \text{the smallest starting value for a hedging strategy that delivers at least the option payoff at maturity, so long as the prediction (1.2) is not violated.} \tag{1.3}$$

Similarly, B is the largest starting value that delivers a payoff not exceeding the option value at expiration.

The main result of Mykland (2000, 2001) is that when an option (vanilla or exotic) matures at a fixed time T, subject to some conditions,

$$A = \sup_{P^*} E^* \left(\exp\{-\int_0^T r_u du\} \times \text{ option payoff} \right), \tag{1.4}$$

where the supremum is over all risk neutral probabilities P^* that allocate probability one to the set (1.2). We emphasize again that r_t and σ_t can be random in any manner whatsoever. See Mykland (2000, 2001) for technical details.

A similar line of inquiry has been pursued by Avellaneda et al. (1995); Lyons (1995); Bergman (1995); and Frey (2000), with varying types of models and bounds. An important feature of the technology is that the trading strategy remains valid for probability distributions for which (1.2) holds with probability less than one, see Mykland (2001).

As illustrated in Table 1 in Sect. 3.1 of Mykland (2001), bounds of type (1.2) (or (2.11) below) are particularly useful for dealing with European style options, while some of the other approaches may be more useful for other types of instruments. Closest to our own work is Bergman (1995), and Frey (2000). We comment on the connection to the former at the end of Sect. 3, and to the latter at the end of the introduction.

It should also be noted that the problem we discuss is different from the superhedging problem discussed in Cvitanić and Karatzas (1992, 1993); El Karoui and Quenez (1995); Eberlein and Jacod (1997); Karatzas (1996); Karatzas and Kou (1996); Kramkov (1996); and Föllmer and Leukert (1999, 2000). These papers discuss the case where the “natural” probability distribution P is known, but the risk neutral one P^* is not. (For further references to this problem, see Mykland 2000). We also emphasize that one is not bound to use the interval specification (1.2). A general construction is given in our earlier papers.

There are two different uses that one can make of hedging strategies based on intervals. On the one hand, obviously, one can actually use these strategies when the intervals are narrow enough. On the other hand, and perhaps more importantly in most circumstances, the intervals provide an indication of how wrong things can go, and thus how much contingency reserves should be allocated to a portfolio. One can then, if one wants, hedge according to a less conservative scheme, such as the one in Hofmann et al. (1992), while using our bounds to make sure that the process is under control. The interval limits (evolving over time) will give the comfort that at any given time one can switch to a conservative strategy.

In other words, the technology advocated here is to a great extent a risk management tool. It is particularly appropriate as such, because it is much easier to make assessments of the type (1.2) than assessments involving which model one should use to describe the market.

A major problem with a methodology that involves intervals for prices is that these can, in many circumstances, be too wide to be useful. There is scope, however, for narrowing these intervals by hedging in auxiliary securities, such as zero coupon bonds and market traded derivatives. The purpose of this paper is to study how this can be implemented for European options. A general framework is developed in Sect. 2. In order to give a concise illustration, we show how to interpolate call options in Sect. 3. As we shall see, this interpolation substantially lowers the upper interval level A from (1.4).

Similar work with different models has been carried out by Bergman (1995), and we return to the connection at the end of Sect. 3. Our reduction of the option value to an optimal stopping problem (Theorem 1 and 2) mirrors the development in Frey (2000). This paper uses the bounds of Avellaneda et al. (cf. Assumption 3 (p. 166) in Frey's paper; the stopping result is Theorem 2.4, p. 167). It goes farther than the present paper in that it also considers certain types of non-European options. See also Frey and Sin (1999).

2 Interpolating European payoffs

For simplicity, consider a stock S_t and a money market bond β_t. The stock pays no dividends. Consider a European payoff $f(S_T)$ to be made at time T. Note that the standard Black-Scholes (1973)-Merton (1973) price at time t can be written $C(S_t, \sigma^2(T-t), r(T-t); f)$, where

$$C(S, \Xi, R; f) = \exp(-R) E f(S \exp(R - \Xi/2 + \sqrt{\Xi} Z)), \tag{2.1}$$

where Z is standard normal (see, for example, Ch. 6 of Duffie 1996).

On the other hand, to illustrate the general methodology from Mykland (2000, 2001), for f either convex or concave, the price A from (1.4) at time 0 will have the form

$$A \;=\; C(S_0, \Xi^{\pm}, 0; h), \tag{2.2}$$

where $h(s) = \sup_{R^- \le R \le R^+} \exp\{-R\} f(\exp\{R\}s)$, and where the $\pm$ on Ξ depends on whether f is convex (+) or concave (−). This is seen by the same methods as those used to prove Theorem 1 below. Note that when f is convex or concave, then so is h.

This starting value is flawed since it does not respect put-call parity (see p. 167 in Hull 1997). To remedy the situation, we now also introduce a zero coupon treasury bond Λ_t. This bond matures with the value one dollar at the time T which is also the expiration date of the European option.

We can now incorporate the uncertainty due to interest rates as follows. Assuming the constraints (1.2), we form the auxiliary function

$$h(s, \lambda; f) = \sup_{R^- \le R \le R^+} \exp\{-R\}[f(\exp\{R\}s) - \lambda] + \lambda \Lambda_0. \tag{2.3}$$

Our result is now that the price for the dynamic hedge equals the best price for a static hedge in Λ_t and a dynamic one in S_t, and that it has the form of the price of an American option.

Theorem 1 *Under the assumptions above, if one hedges in S_t and Λ_t, the quantity A in (1.3) has the form*

$$A(f) = \inf_{\lambda} \sup_{\tau} \widetilde{E} h(\widetilde{S}_\tau, \lambda; f) \tag{2.4}$$

where:

$$\widetilde{P} : d\widetilde{S}_t = \widetilde{S}_t d\widetilde{W}_t,\ \widetilde{S}_0 = S_0 \tag{2.5}$$

and τ is any stopping time between Ξ^- and Ξ^+.

We emphasize that what was originally cumulative volatilities (Ξ^-, Ξ^+) have now become measures of time when computing (2.4). This is because of the Dambis (1965)/Dubins-Schwartz (1965) time change, which leads to time being measured on the volatility scale.

Remark 1 Note that in Theorem 1, and in the similar results below, the optimization involving R and λ can be summarized by replacing (2.4) with $A(f) = \sup_\tau \widetilde{E} g(\widetilde{S}_\tau; f)$, where $g(s; f)$ is the supremum of $Eh(s, \lambda; f)$ over (random variables) $R \in [R^-, R^+]$, subject to $E(\exp\{-R\}) = \Lambda_0$. R becomes a function of s, which in the case of convex f will take values R^- and R^+. This type of development is further pursued in Sect. 4 of Mykland (2000).

As in (2.2), if f is convex or concave, then so is the h in (2.3). In other words, since convex functions of martingales are submartingales, and concave ones are supermartingales (see, for example, Karatzas and Shreve 1991, Proposition I.3.6, p. 13), the result in Theorem 1 simplifies in those cases:

$$\begin{aligned} f \text{ convex: } & A(f) = \inf_{\lambda} \widetilde{E} h(\widetilde{S}_{\Xi^+}, \lambda; f), \text{ and} \\ f \text{ concave: } & A(f) = \inf_{\lambda} \widetilde{E} h(\widetilde{S}_{\Xi^-}, \lambda; f), \end{aligned} \tag{2.6}$$

both of which expressions are analytically computable. (The expression for convex f is given in Mykland 2000), where this was studied as a special case).

We can then move on to the case of auxiliary market traded European derivatives. We then also suppose that there are p such derivatives $V_t^{(i)}$ $(i = 1, ..., p)$ whose payoffs are $f_i(S_T)$ at time T. Again, it is the case that the price for the dynamic hedge equals the best price for a static hedge in the auxiliary securities, with a dynamic one in S_t only:

Theorem 2 *Under the assumptions above, if one hedges in S_t, Λ_t, and the $V_t^{(i)}$ $(i = 1, ..., p)$, the quantity A in (1.3) has the form*

$$A(f; f_1, ..., f_p) = \inf_{\lambda_1 ..., \lambda_p} A(f - \lambda_1 f_1 - ...\lambda_p f_p) + \sum_{i=1}^{p} \lambda_i V_0^{(i)}, \tag{2.7}$$

where $A(f - \lambda_1 f_1 - ...\lambda_p f_p)$ is as given by (2.4)-(2.5).

Remark 2 Bid prices are formed similarly. In Theorem 1,

$$B(f) \;=\; \sup_{\lambda} \inf_{\tau} \widetilde{E} h(\widetilde{S}_\tau, \lambda; f), \tag{2.8}$$

and in Theorem 2,

$$B(f; f_1, ..., f_p) = \sup_{\lambda_1 ..., \lambda_p} B(f - \lambda_1 f_1 - ...\lambda_p f_p) + \sum_{i=1}^{p} \lambda_i V_0^{(i)}. \tag{2.9}$$

Note the relationship between the two types of prices:

$$B(f) = -A(-f) \quad \text{and } B(f; f_1, ..., f_p) = -A(-f; f_1, ..., f_p). \tag{2.10}$$

The following elegant way of dealing with uncertain interest was first encountered by this author in the work of El Karoui et al. (1998).

Remark 3 A special case which falls under the above is one where one has a prediction interval for the volatility of the future S^* on S. Set $S_t^* = S_t/\Lambda_t$, and replace Eq. (1.1) by $dS_t^* = \mu_t S_t^* dt + \sigma_t S_t^* dW_t^*$. S^* is then the value of S in *numeraire* (unit of account) Λ, and the interest rate is zero in this numeraire. By *numeraire invariance* (see, for example, Chapter 6.B (pp. 102–103) of Duffie 1996), one can now treat the problem in this unit of account. If one has an interval

$$\Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-, \tag{2.11}$$

this is therefore the same as the problem posed in the form (1.2), with $R^- = R^+ = 0$. There is no mathematical difference, but note that (2.11) is now an interval for the volatility of the future S^* rather than the actual stock price S.

Still with numeraire Λ, The Black-Sholes price is $C(S_0, \Xi, -\log \Lambda_0; f)/\Lambda_0 = C(S_0^*, \Xi, 0; f)$. In this case, h (from (2.3)) equals f. Theorems 1-3, Algorithm 1, and Corollary 4 go through unchanged. For example, Eq. (2.4) becomes (after reconversion to dollars)

$$A(f) \;=\; \Lambda_0 \sup_{\tau} \widetilde{E} f(\widetilde{S}_\tau). \tag{2.12}$$

where the initial value in (2.5) is $\widetilde{S}_0 = S_0^* = S_0/\Lambda_0$.

3 The case of European calls

To simplify our discussion, we shall in the following assume that the short term interest rate r is known, so that $R^+ = R^- = rT$. This case also covers the case of the bounds described in Remark 3 (with $r = 0$). The case of unknown interest rate with bounds (1.2) (but no market traded options) has been partially explored in Mykland (2000). We focus here on the volatility only since this seems to be the foremost concern of most traders. In other words, our prediction interval is

$$\Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-. \tag{3.1}$$

Consider, therefore, the case where one wishes to hedge an option with payoff $f_0(S_T)$, where f_0 is (non strictly) convex. We suppose that there are, in fact, market traded call options $V_t^{(1)}$ and $V_t^{(2)}$ with strike prices K_1 and K_2. We suppose that $K_1 < K_2$, and set $f_i(s) = (s - K_i)^+$.

From Theorems 1-2, the price A at time 0 for payoff $f_0(S_T)$ is

$$A(f_0; f_1, f_2) = \inf_{\lambda_1,\lambda_2} \sup_{\tau} \widetilde{E}(h - \lambda_1 h_1 - \lambda_2 h_2)(\widetilde{S}_\tau) + \sum_{i=1}^{2} \lambda_i V_0^{(i)}, \tag{3.2}$$

where, for $i = 1, 2$, $h_i(s) = \exp\{-rT\} f_i(\exp\{rT\}s) = (s - K_i')^+$, with $K_i' = \exp\{-rT\}K_i$.

We now give an algorithm for finding A.

For this purpose, let $C(S, \Xi, R, K)$ be as defined in (2.1) for $f(s) = (s - K)^+$ (in other words, the Black-Scholes-Merton price for a European call with strike price K). Also define, for $\Xi \leq \widetilde{\Xi}$,

$$\widetilde{C}(S, \Xi, \widetilde{\Xi}, K, \widetilde{K}) \;=\; \widetilde{E}((\widetilde{S}_\tau - \widetilde{K})^+ \mid S_0 = S), \tag{3.3}$$

where τ is the minimum of $\widetilde{\Xi}$ and the first time after Ξ that $\widetilde{S}_t$ hits K. An analytic expression for (3.3) is given as Eq. (A.7) in the Appendix.

Algorithm 1

(i) Find the implied volatilities Ξ_i^{impl} of the options with strike price K_i. In other words, $C(S_0, \Xi_i^{\text{impl}}, rT, K_i) = V_0^{(i)}$.

(ii) If $\Xi_1^{\text{impl}} < \Xi_2^{\text{impl}}$, set $\Xi_1 = \Xi_1^{\text{impl}}$, but adjust Ξ_2 to satisfy $C(S_0, \Xi_1^{\text{impl}}, \Xi_2, K_1', K_2') = V_0^{(2)}$. If $\Xi_1^{\text{impl}} > \Xi_2^{\text{impl}}$, do the opposite, in other words, keep $\Xi_2 = \Xi_2^{\text{impl}}$, and adjust Ξ_1 to satisfy $C(S_0, \Xi_2^{\text{impl}}, \Xi_1, K_2', K_1') = V_0^{(1)}$. If $\Xi_1^{\text{impl}} = \Xi_2^{\text{impl}}$, leave them both unchanged, i.e., $\Xi_1 = \Xi_2 = \Xi_1^{\text{impl}} = \Xi_2^{\text{impl}}$.

(iii) Define a stopping time τ as the minimum of Ξ^+, the first time $\widetilde{S}_t$ hits K_1' after Ξ_1, and the first time $\widetilde{S}_t$ hits K_2' after Ξ_2. Then

$$A(f_0; f_1, f_2) \;=\; \widetilde{E} h_0(\widetilde{S}_\tau). \tag{3.4}$$

Note in particular that if f_0 is also a call option, with strike K_0, and still with the convention $K_0' = \exp\{-rT\}K_0$, one obtains

$$A = \widetilde{E}(\widetilde{S}_\tau - K_0')^+. \tag{3.5}$$

This is the sense in which one could consider the above an interpolation or even extrapolation: the strike prices K_1 and K_2 are given, and K_0 can now vary.

Theorem 3 *Suppose that $\Xi^- \leq \Xi_1^{\text{impl}}, \Xi_2^{\text{impl}} \leq \Xi^+$. Then the A found in Algorithm 1 coincides with the one given by (3.2). Furthermore, for $i = 1, 2$,*

$$\Xi_i^{\text{impl}} \leq \Xi_i. \tag{3.6}$$

Note that the condition $\Xi^- \leq \Xi_1^{\text{impl}}, \Xi_2^{\text{impl}} \leq \Xi^+$ must be satisfied to avoid arbitrage, assuming one believes the bound (3.1). Also, though Theorem 3 remains valid, no-arbitrage considerations impose constraints on Ξ_1 and Ξ_2, as follows.

Corollary 4 *Assume $\Xi^- \leq \Xi_1^{\text{impl}}, \Xi_2^{\text{impl}} \leq \Xi^+$. Then Ξ_1 and Ξ_2 must not exceed Ξ^+. Otherwise there is arbitrage under the condition (3.1).*

We prove the algorithm and the corollary in the Appendix. Note that $\widetilde{C}(S, \Xi, \widetilde{\Xi}, K, \widetilde{K})$ in (3.3) is a down-and-out type call for $\widetilde{K} \geq K$, and can be rewritten as an up-and-out put for $\widetilde{K} < K$, and is hence obtainable in closed form – cf. Eq. (A.7) in the Appendix. A in (3.5) has a component which is on the form of a double barrier option, so the analytic expression (which can be found using the methods in Ch. 2.8, p. 94–103 in Karatzas and Shreve 1991) will involve an infinite sum (as in *ibid*, Proposition 2.8.10, p. 98). See also Geman and Yor (1996) for analytic expressions. Simulations can be carried out using theory in Asmussen et al. (1995); and Simonsen (1997).

The pricing formula does not explicitly involve Ξ^-. It is implicitly assumed, however, that the implied volatilities of the two market traded options exceed Ξ^-. Otherwise, there would be arbitrage opportunities. This, obviously, is also the reason why one can assume that $\Xi_i^{\text{impl}} \leq \Xi^+$ for both i.

How does this work in practice? We consider an example scenario in Figs. 1 and 2. We suppose that market traded calls are sparse, so that there is nothing between $K_1 = 100$ (which is at the money), and $K_2 = 160$. Figure 1 gives implied volatilities of A as a function of the upper limit Ξ^+. Figure 2 gives the implied volatilities as a function of K_0. As can be seen from the plots, the savings over using volatility Ξ^+ are substantial.

All the curves in Fig. 1 have an asymptote corresponding to the implied volatility of the price $A_{\text{crit}} = \lambda_1^{(0)} V_0^{(1)} + (1 - \lambda_1^{(0)}) V_0^{(2)}$, where $\lambda_1^{(0)} = (K_2 - K_0)/(K_2 - K_1)$. This is known as the Merton bound, and holds since, obviously, $\lambda_1^{(0)} S_t^{(1)} + (1 - \lambda_1^{(0)}) S_t^{(2)}$ dominates the call option with strike price K_0, and is the cheapest linear combination of $S_t^{(1)}$ and $S_t^{(2)}$ with this property. In fact, if one denotes as A_{Ξ^+} the quantity from (3.5), and if the Ξ_i^{impl} are kept fixed, it is easy to see that, for (3.5),

$$\lim_{\Xi^+ \to +\infty} A_{\Xi^+} = A_{\text{crit}}. \tag{3.7}$$

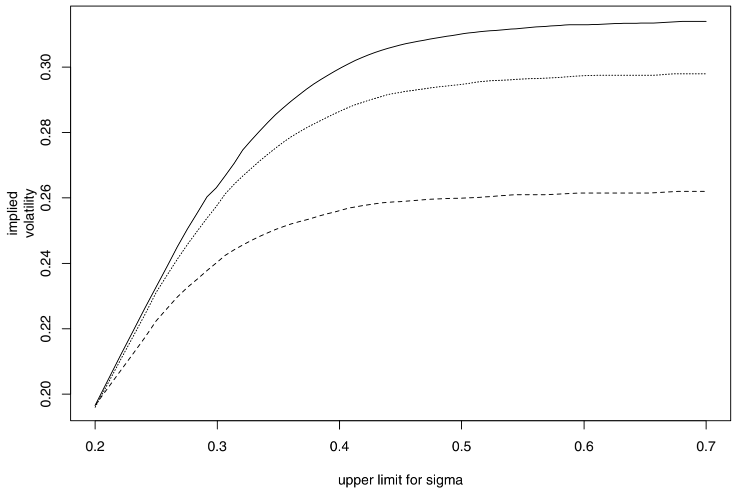

Fig. 1. Effect of interpolation: Implied volatilities for interpolated call options as a function of the upper limit of the prediction interval. We consider various choices of strike price K_0 (from *top* to *bottom*: K_0 is 130, 120 and 110) for the option to be interpolated. The options that are market traded have strike prices $K_1 = 100$ and $K_2 = 160$. The graph shows the implied volatility of the options price A (σ_{impl} given by $C(S_0, \sigma^2_{\mathrm{impl}}, rT, K_0) = A$ as a function of $\sqrt{\Xi^+}$. We are using square roots as this is the customary reporting form. The other values defining the graph are $S_0 = 100, T = 1$ and $r = 0.05$, and $\sqrt{\Xi_1^{\mathrm{impl}}} = \sqrt{\Xi_2^{\mathrm{impl}}} = 0.2$. The asymptotic value of each curve corresponds to the Merton bound for that volatility

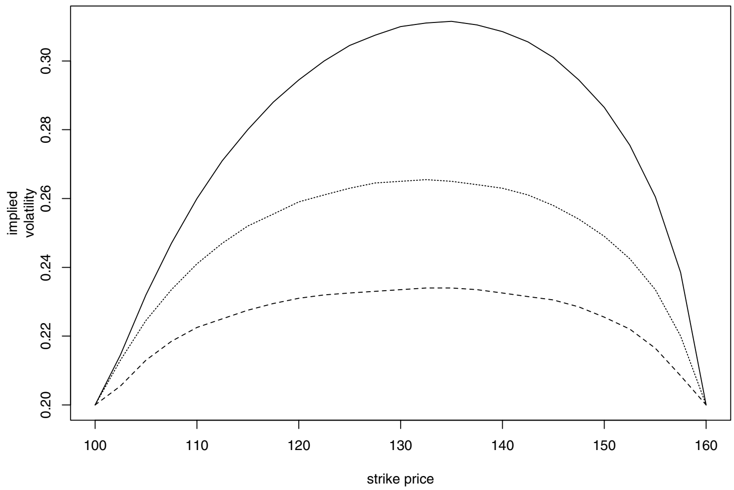

Fig. 2. Effect of interpolation: implied volatilities for interpolated call options as a function of the strike price K_0 for the option to be interpolated. We consider various choices of maximal volatility values $\sqrt{\Xi^+}$ (from *top* to *bottom*: $\sqrt{\Xi^+}$ is 0.50, 0.40 and 0.25). Other quantities are as in Fig. 1. Note that the curve for $\sqrt{\Xi^+} = 0.50$ is graphically indistinguishable from that of the Merton bound

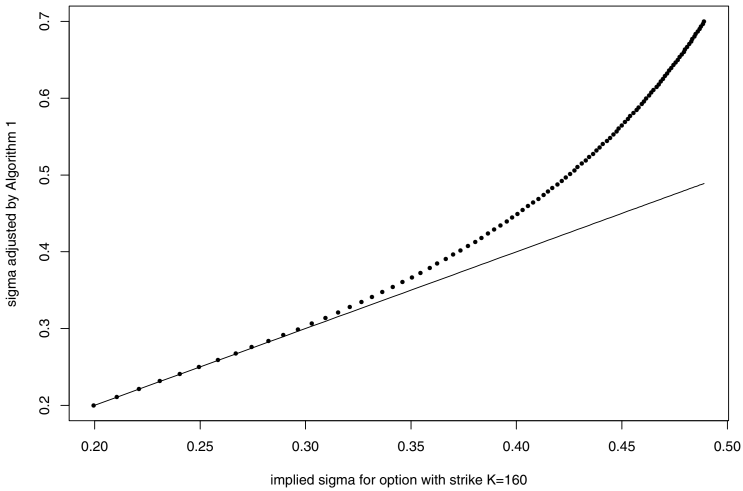

Fig. 3. $\widetilde{C}$: $\sqrt{\Xi_2}$ as a function of $\sqrt{\Xi_2^{\mathrm{impl}}}$, for fixed $\sqrt{\Xi_1^{\mathrm{impl}}} = \sqrt{\Xi_2} = 0.2$. A *diagonal line* is added to highlight the functional relationship

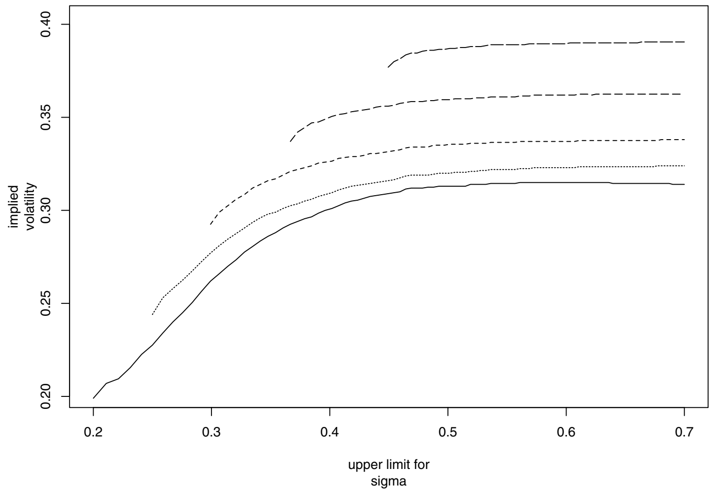

Fig. 4. Implied volatility for interpolated call option with strike price $K_0 = 140$, as the upper bound $\sqrt{\Xi^+}$ varies. The curves assume $\sqrt{\Xi_1^{\mathrm{impl}}} = 0.2$ and, in ascending order, correspond to $\sqrt{\Xi_2^{\mathrm{impl}}} = 0.2,\ 0.25,\ 0.3,\ 0.35$ and 0.4. The starting point for each curve is the value $\sqrt{\Xi^+}$ (on the x axis) so that the no-arbitrage condition of Corollary 4 is not violated. As in Fig. 1, the asymptotic value of each curve corresponds to the Merton bound for that volatility

Figures 1 and 2 presuppose that the implied volatility of the two market traded options are the same ($\sqrt{\Xi_1^{\mathrm{impl}}} = \sqrt{\Xi_2^{\mathrm{impl}}} = 0.2$). To see what happens when the out of the money option increases its implied volatility, we fix $\sqrt{\Xi_1^{\mathrm{impl}}} = 0.2$, and we show in Fig. 3 the plot of $\sqrt{\Xi_2}$ as a function of $\sqrt{\Xi_2^{\mathrm{impl}}}$. Also, we give in Fig. 4 the implied volatilities for the interpolated option (3.5) with strike price $K_0 = 140$. We see that except for high $\sqrt{\Xi_2^{\mathrm{impl}}}$, there is still gain by a constraint on the form (3.1).

It should be noted that there is similarly between the current paper and the work by Bergman (1995). This is particularly so in that he finds an arbitrage relationship between the value of two options (see his Sect. 3.2 pp. 488–494, and in particular Proposition 4). Our development, similarly, finds an upper limit for the price of a third option given two existing ones. As seen in Corollary 4, it can also be applied to the relation between two options only.

The similarly, however, is mainly conceptual, as the model assumptions are substantially different. An interest rate interval (Bergman's Eqs. (1)–(2) on p. 478) is obtained by differentiating between lending and borrowing rates (as also in Cvitanić and Karatzas 1993), and the stock price dynamic is given by differential Eqs. (3)–(4) on p. 479. This is in contrast to our assumptions (1.2) or in Remark 3. It is, therefore, hard to compare Bergman's and our results in other than conceptual terms.

4 Conclusion

We have shown in the above that the interpolation of options can substantially reduce the length of intervals for prices that are generated under uncertainty in the predicted volatility and interest rates. It would be natural to extend the approach to the case of several securities, and in particular confront the common reality that the volatility itself is quite well pinned down, whereas correlations are not. An even more interesting question is whether this kind of nonparametrics can be used in connection with the interest rate term structure, where the uncertainty about models is particularly acute.

We have in the entire paper treated the intervals for volatilities and interest rates as given, by the statistician in the next office, presumably. One may also wish look into ways of integrating these processes more closely. For example, if half time to expiration has come and it looks like the prediction interval is off in one direction or the other, how can one handle that without prejudicing the unconditional probability of carrying out a successful hedge? There are a number of interesting questions here.

Appendix

Poof of Theorem 1. By (1.4),

$$A(f) = \sup_{Q^* \in \mathcal{Q}^*} E_{Q^*} \exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*)$$

where $\mathcal{Q}^*$ is the set of all probability distributions Q^* so that (1.2) is satisfied, so that $\Lambda_t^* (= \exp\{-\int_0^t r_u du\}\Lambda_t)$ is a martingale, and so that $dS_t^* = \sigma_t S_t^* dW_t$, for given S_0. For a given $Q^* \in \mathcal{Q}^*$, define $Q^{(1)}$, also in $\mathcal{Q}^*$, by letting $v > 1$, $\sigma_t^{\text{new}} = \sigma_{vt}$ for $vt \leq T$ and zero thereafter until T. Whereas we let $r_t^{\text{new}} = 0$ until T/v, and thereafter let $r_t^{\text{new}} = r_{(vt-T)/(v-1)}$.

Since the value of $\exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*)$ does not depend on the order in which the interest and the volatility accrue, and since $E \exp\{-\int_0^T r_u du\} = \Lambda_0$ for all distributions in $\mathcal{Q}^*$, one obtains

$$\begin{aligned}
&E_{Q^*} \exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*) \\
&= E_{Q^{(1)}} \exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*) \\
&\qquad + \lambda(\Lambda_0 - E_{Q^{(1)}} \exp\{-\int_0^T r_u du\}) \\
&\leq E_{Q^{(1)}} h(S_T^*, \lambda; f) \\
&= E_{Q^*} h(S_T^*, \lambda; f)
\end{aligned}$$

for all λ, by definition of h. The first inequality is valid since under $Q^{(1)}$, one can take the supremum over $\int_0^T r_u du$ after observing the value of S_T^*. Cf. also Remark 1.

By using the Dambis (1965)/Dubins-Schwarz (1965) time change (see, e.g., Karatzas and Shreve 1991, p. 173–179), we can write $\widetilde{S}_{\tau_t} = S_t^*$, where the τ_t are stopping times so that

$$\int_0^t \sigma_u^2 du = \tau_t.$$

This gives $\widetilde{S}_u$ the dynamic (2.5), with the constraint $\Xi^- \leq \int_0^T \sigma_u^2 du \leq \Xi^+$ translated into $\Xi^- \leq \tau_T \leq \Xi^+$.

Hence, continuing the above,

$$\begin{aligned}
E_{Q^*} \exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*) &\leq E_{Q^*} h(S_T^*, \lambda; f) \\
&\leq \sup_{\Xi^- \leq \tau_T \leq \Xi^+} \widetilde{E} h(\widetilde{S}_{\tau_T}, \lambda; f).
\end{aligned}$$

Hence we can take the infimum over λ, to get that $A(f)$ in (2.4) is an upper bound.

To see that this is also a lower bound, define $Q^{(2)}$, also in $\mathcal{Q}^*$, by still letting the volatility be σ_t^{new}, but now letting $r_t^{(2)} = 0$ for $t < T/v$, and $r_t^{(2)} = Rv/T(1-v)$, where R maximizes the conditional expected value of the right hand side of (2.3) given $s = S_T^*$ and subject to $E \exp\{-R\} = \Lambda_0$. We obtain directly that

$$E_{Q^{(2)}} \exp\{-\int_0^T r_u du\} f(\exp\{\int_0^T r_u du\} S_T^*) = \inf_\lambda E_{Q^*} h(S_T^*, \lambda; f)$$

By again using the Dambis (1965)/Dubins-Schwarz (1965) time change, it is now easy to see that $A(f)$ in (2.4) is the supremum over distributions in $\mathcal{Q}^*$. Theorem 1 follows. □

Proof of Theorem 2 This result follows in a similar way to the proof of Theorem 1, with the modification that $\mathcal{Q}^*$ is now the set of all probability distributions Q^* so that (1.2) is satisfied, so that Λ_t^* *and* the $V_t^{(i)*}$ $(i = 1, ..., p)$ are martingales, and so that $dS_t^* = \sigma_t S_t^* dW_t$, for given S_0. □

Before we proceed to the proof of Theorem 3, let us establish the following set of inequalities for $\Xi < \widetilde{\Xi}$,

$$C(S, \Xi, R, K_2) < \widetilde{C}(S, \Xi, \widetilde{\Xi}, K_1', K_2') < C(S, \widetilde{\Xi}, R, K_2). \qquad \text{(A.1)}$$

The reason for this is that $\widetilde{C}(S, \Xi, \widetilde{\Xi}, K_1', K_2') = \widetilde{E}((\widetilde{S}_\tau - K_2')^+)$ is nondecreasing in both Ξ and $\widetilde{\Xi}$, since $\widetilde{S}$ is a martingale and $x \to x^+$ is convex, and also that $\widetilde{C}(S, \Xi, \Xi, K_1', K_2') = C(S, \Xi, 0, K_2') = C(S, \Xi, R, K_2)$. The inequalities are obviously strict otherwise.

Proof of Theorem 3 (and Algorithm 1) We wish to find (3.2). First fix λ_1 and λ_2, in which case we are seeking $\sup_\tau \widetilde{E} h_{\lambda_1,\lambda_2}(\widetilde{S}_\tau)$, where $h_{\lambda_1,\lambda_2} = h_0 - \lambda_1 h_1 - \lambda_2 h_2$. This is because the $V_0^{(i)}$ are given. We recall that h_0 is (non strictly) convex since f_0 has this property, and that $h_i(s) = (s - K_i')^+$. It follows that h_{λ_1,λ_2} is convex except at points $s = K_1'$ and $= K_2'$.

Since $\widetilde{S}_t$ is a martingale, $h_{\lambda_1,\lambda_2}(\widetilde{S}_t)$ is therefore a submartingale so long as $\widetilde{S}_t$ does not cross K_1' or K_2' (see Proposition I.3.6, p. 13 in Karatzas and Shreve 1991). It follows that if τ_0 is a stopping time, $\Xi^- \leq \tau_0 \leq \Xi^+$, and we set

$$\tau = \inf\{\, t \geq \tau_0 : \widetilde{S}_t = K_1' \text{ or } K_2' \,\} \wedge \Xi^+,$$

then $\widetilde{E} h_{\lambda_1,\lambda_2}(\widetilde{S}_{\tau_0}) \leq \widetilde{E} h_{\lambda_1,\lambda_2}(\widetilde{S}_\tau)$. It follows that the only possible optimal stopping points would be $\tau = \Xi^+$ and τs for which $\widetilde{S}_\tau = K_i'$ for $i = 1, 2$.

Further inspection makes it clear that the rule must be on the form given in part (iii) of the algorithm, but with Ξ_1 and Ξ_2 as yet undetermined. This comes from standard arguments for American options (see Karatzas 1988; Myneni 1992, and the references therein), as follows. Define the *Snell envelope* for h_{λ_1,λ_2} by

$$\mathrm{SE}(s, \Xi) = \sup_{\Xi \leq \tau \leq \Xi^+} \widetilde{E}(h_{\lambda_1,\lambda_2}(\widetilde{S}_\tau) \mid S_\Xi = s).$$

The solution for American options is then that

$$\tau \;=\; \inf\{\, \xi \geq \Xi^- \;:\; \mathrm{SE}(\widetilde{S}_\xi, \xi) \;=\; h_{\lambda_1,\lambda_2}(\widetilde{S}_\xi) \,\}$$

Inspection of the preceding formula yields that $\tau = \tau_1 \wedge \tau_2$, where

$$\begin{aligned}\tau_i &= \inf\{\, \xi \geq \Xi^- \;:\; \{\, \mathrm{SE}(\widetilde{S}_\xi, \xi) \;=\; h_{\lambda_1,\lambda_2}(\widetilde{S}_\xi) \,\} \cap \{\, \widetilde{S}_\xi = K_i' \,\} \,\} \wedge \Xi^+ \\ &= \inf\{\, \xi \geq \Xi^- \;:\; \{\, \mathrm{SE}(K_i', \xi) \;=\; h_{\lambda_1,\lambda_2}(K_l') \,\} \cap \{\, \widetilde{S}_\xi = K_i' \,\} \,\} \wedge \Xi^+ \\ &= \inf\{\, \xi \geq \Xi_i \;:\; \widetilde{S}_\xi = K_i' \,\} \wedge \Xi^+,\end{aligned}$$

where $\Xi_i = \inf\{ \xi \geq \Xi^- \ : \ \mathrm{SE}(K_i', \xi) \ = \ h_{\lambda_1,\lambda_2}(K_i') \} \wedge \Xi^+$.

Since the system in linear in λ_1 and λ_2, and in analogy with the discussion in Remark 1, it must be the case that

$$\widetilde{E}(\widetilde{S}_\tau - K_i')^+ = V_0^{(i)} \text{ for } i = 1, 2. \tag{A.2}$$

Hence the form of A given in part (iii) of the algorithm must be correct, and one can use (A.2) to find Ξ_1 and Ξ_2. Note that the left hand side of (A.2) is continuous and increasing in Ξ_1 and Ξ_2, (again since $\widetilde{S}$ is a martingale and $x \to x^+$ is convex). Combined with our assumption in Theorem 3 that $\Xi^- \leq \Xi_1^{\mathrm{impl}}, \Xi_2^{\mathrm{impl}} \leq \Xi^+$, we are assured that (A.2) has solutions Ξ_1 and Ξ_2 in $[\Xi^-, \Xi^+]$, if necessary by increasing Ξ^+.

Let (Ξ_1, Ξ_2) be a solution for (A.2) (we have not yet decided what values they take, or even that they are in the interval $[\Xi^-, \Xi^+]$).

Suppose first that $\Xi_1 < \Xi_2$.

It is easy to see that

$$\widetilde{E}[(\widetilde{S}_\tau - K_1')^+ \mid \widetilde{S}_{\Xi_1}] = (\widetilde{S}_{\Xi_1} - K_1')^+. \tag{A.3}$$

This is immediate when $\widetilde{S}_{\Xi_1} \leq K_1'$; in the opposite case, note that $(\widetilde{S}_\tau - K_1')^+ = \widetilde{S}_\tau - K_1'$ when $\widetilde{S}_{\Xi_1} > K_1'$, and one can then use the martingale property of $\widetilde{S}_t$. Taking expectations in (A.3) yields from (A.2) that Ξ_1 must be the implied volatility of the call with strike price K_1.

Conditioning on $\mathcal{F}_{\Xi_2}$ is a little more complex. Suppose first that $\inf_{\Xi_1 \leq t \leq \Xi_2} \widetilde{S}_t > K_1'$. This is equivalent to $\tau > \Xi_2$, whence

$$\widetilde{E}[(\widetilde{S}_\tau - K_2')^+ \mid \mathcal{F}_{\Xi_2}] = (\widetilde{S}_{\Xi_2} - K_2')^+,$$

as in the previous argument (separate into the two cases $\widetilde{S}_{\Xi_2} \leq K_2'$ and $\widetilde{S}_{\Xi_2} > K_2'$). Hence, incorporating the case where $\tau \leq \Xi_2$, we find that

$$\widetilde{E}(\widetilde{S}_\tau - K_2')^+ = \widetilde{E}(\widetilde{S}_{\Xi_2 \wedge \tau} - K_2')^+,$$

thus showing that Ξ_2 can be obtained from $\widetilde{C}(S_0, \Xi_1^{\mathrm{impl}}, \Xi_2, K_1', K_2') = V_0^{(2)}$. In consequence, from the left hand inequality in (A.1),

$$\begin{aligned} C(S_0, \Xi_1^{\mathrm{impl}}, rT, K_2) &< \widetilde{C}(S_0, \Xi_1^{\mathrm{impl}}, \Xi_2, K_1', K_2') \\ &= V_0^{(2)} = C(S_0, \Xi_2^{\mathrm{impl}}, rT, K_2) \end{aligned}$$

Since, for call options, $C(S, \Xi, R, K_2)$ is increasing in Ξ, it follows that $\Xi_2^{\mathrm{impl}} > \Xi_1^{\mathrm{impl}}$.

Hence, under the assumption that $\Xi_1 < \Xi_2$, Algorithm 1 produces the right result.

The same arguments apply in the cases $\Xi_1 > \Xi_2$ and $\Xi_1 = \Xi_2$, in which cases, respectively, $\Xi_1^{\mathrm{impl}} > \Xi_2^{\mathrm{impl}}$ and $\Xi_1^{\mathrm{impl}} = \Xi_2^{\mathrm{impl}}$. Hence, also in these cases, Algorithm 1 provides the right solution.

Hence the solution to (A.2) is unique and is given by Algorithm 1.

The inequality in (3.6) follows because the adjustment in (ii) increases the value of of the Ξ_i that is adjusted. This is because of the rightmost inequality in (A.1).

The result follows. □

An analytic expression for Eq. (3.3). To calculate the expression (3.3), note first that

$$\widetilde{C}(S,\Xi,\widetilde{\Xi},K,\widetilde{K}) \;=\; \widetilde{E}[\widetilde{C}(S_{\Xi},0,\widetilde{\Xi}-\Xi,K,\widetilde{K})|S_0=S]$$

We therefore first concentrate on the expression for $\widetilde{C}(s,0,T,K,\widetilde{K})$. For $K<\widetilde{K}$, this is the price of a down and out call, with strike $\widetilde{K}$, barrier K, and maturity T. We are still under the $\widetilde{P}$ distribution, in other words, $\sigma=1$ and all interest rates are zero. The formula for this price is given on p. 462 in Hull (1997), and because of the unusual values of the parameters, one gets

$$\begin{aligned}\widetilde{C}(s,0,T,K,\widetilde{K}) &= \widetilde{E}((S_T-\widetilde{K})^+|S_0=s)\\ &\quad -\frac{\widetilde{K}}{K}\widetilde{E}((S_T-H)^+|S_0=s)+\frac{\widetilde{K}}{K}(s-H)\end{aligned}$$

for $s>K$, while the value is zero for $s\le K$. Here, $H=K^2/\widetilde{K}$.

Now set

$$D(s,\Xi,\widetilde{\Xi},K,X) \;=\; \widetilde{E}[(S_{\widetilde{\Xi}}-X)^+ I\{S_{\Xi}\ge K\}|S_0=s]$$

and let BS_0 be the Black-Scholes formula for zero interest rate and unit volatility, $BS_0(s,\Xi,X) \;=\; \widetilde{E}[(S_{\Xi}-X)^+|S_0=s]$, in other words,

$$BS_0(s,\Xi,X) \;=\; s\Phi(d_1(s,X,\Xi)) \;-\; X\Phi(d_2(s,X,\Xi)), \tag{A.4}$$

where Φ is the cumulative standard normal distribution, and

$$\begin{aligned}&d_i=d_i(s,X,\Xi)=(\log(s/X)\pm\Xi/2)/\sqrt{\Xi}\text{ where }\pm\text{ is}\\ &\quad +\text{ for }i=1\text{ and }-\text{ for }i=2.\end{aligned} \tag{A.5}$$

Then, for $K<\widetilde{K}$,

$$\begin{aligned}\widetilde{C}(s,\Xi,\widetilde{\Xi},K,\widetilde{K}) &= D(s,\Xi,\widetilde{\Xi},K,\widetilde{K})-\frac{\widetilde{K}}{K}D(s,\Xi,\widetilde{\Xi},K,H)\\ &\quad +\frac{\widetilde{K}}{K}BS_0(s,\Xi,K)+(\widetilde{K}-K)\Phi(d_2(s,K,\Xi)).\end{aligned} \tag{A.6}$$

Similarly, for $K\ge\widetilde{K}$, a martingale argument and the formula on p. 463 in Hull (1997) gives that

$$\begin{aligned}&\widetilde{C}(s,0,T,K,\widetilde{K})\\ &=s-\widetilde{K}\;+\;\text{ value of up and out put option with strike }\widetilde{K}\text{ and barrier }K\\ &=\widetilde{E}((S_T-\widetilde{K})^+|S_0=s)\;-\;\text{ value of up and in put option with strike }\widetilde{K}\\ &\quad\text{and barrier }K\\ &=\widetilde{E}((S_T-\widetilde{K})^+|S_0=s)\;-\;\frac{\widetilde{K}}{K}\widetilde{E}((S_T-H)^+|S_0=s)\text{ for }s<K.\end{aligned}$$

On the other hand, obviously, for $s \geq K$, $\widetilde{C}(s,0,T,K,\widetilde{K}) = (s-\widetilde{K})$ by a martingale argument.

Hence, for $K \geq \widetilde{K}$, we get

$$\begin{aligned}&\widetilde{C}(s,\Xi,\widetilde{\Xi},K,\widetilde{K}) \\ &= BS_0(s,\widetilde{\Xi},\widetilde{K}) - \frac{\widetilde{K}}{K}BS_0(s,\widetilde{\Xi},H) - D(s,\Xi,\widetilde{\Xi},K,\widetilde{K}) \\ &+\frac{\widetilde{K}}{K}D(s,\Xi,\widetilde{\Xi},K,H) + BS_0(s,\Xi,K) + (K-\widetilde{K})\Phi(d_2(s,K,\Xi)).\end{aligned} \tag{A.7}$$

The formula for D is

$$\begin{aligned}D(s,\Xi,\widetilde{\Xi},K,X) &= s\Phi(d_1(s,X,\widetilde{\Xi}),d_1(s,K,\Xi);A) \\ &\quad - X\Phi(d_2(s,X,\widetilde{\Xi}),d_2(s,K,\Xi);A),\end{aligned} \tag{A.8}$$

where

$$\Phi(x,y;A) \;=\; \text{cumulative bivariate normal c.d.f. with covariance matrix } A \tag{A.9}$$

and A is the matrix with diagonal elements 1 and off diagonal elements ρ,

$$\rho \;=\; \sqrt{\frac{\Xi}{\widetilde{\Xi}}}. \tag{A.10}$$

□

Proof of Corollary 4 It is easy to see that Theorem 3 goes through with $K_1 = K_2$ (in the case where the implied volatilities are the same). Using formula (3.6), we get from Algorithm 3 that

$$A((s-K_0)^+;(s-K_1)^+) = \widetilde{C}(S_0,\Xi_1^{\text{impl}},\Xi^+,K_1',K_1'). \tag{A.11}$$

The result then follows by replacing “0” by “2” in (A.11). □

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