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Rounding Errors and Volatility Estimation

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ABSTRACT
Financial prices are often discretized—with smallest tick size of one cent, for example. Thus prices involve rounding errors. Rounding errors affect the estimation of volatility, and understanding them is critical, particularly when using high frequency data. We study the asymptotic behavior of realized volatility (RV), which is commonly used as an estimator of integrated volatility. We prove the convergence of the RV and scaled RV under varous conditions on the rounding level and the number of observations. A bias-corrected volatility estimator is proposed and an associated central limit theorem is shown. The simulation and empirical results demonstrate that the proposed method can yield substantial statistical improvement. (*JEL*: C02, C13,C14)

KEYWORDS: rounding errors, bias-correction, diffusion process, market microstructure, realized volatility (RV)

High frequency data analysis has received substantial attention in recent years and volatility estimation is a central topic of interest. The primary difficulty in estimating daily volatilities by using high frequency data is the presence of market microstructure noise; notable developments have been made in this area. Zhang et al. (2005); Zhang (2006); Barndorff-Nielsen et al. (2008); and Xiu (2010) proposed and evaluated various volatility estimators that exhibited favorable

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convergence properties, assuming that the microstructure noise was additive, and independent and identically distributed (i.i.d.). Li and Mykland (2007) and Jacod et al. (2009) studied cases in which the market microstructure noise was a combination of additive noise and rounding error. Rosenbaum (2009) proposed a novel volatility estimation approach, using absolute values of the increments when rounding is the only source of the market microstructure noise.

Rounding is a crucial source of market microstructure noise that should not be ignored. Because stocks are traded using discrete price grids, their observations are effectively rounded. In certain cases, particularly when the stock prices are low, rounding can be the main source of market microstructure noise. Figure 1 shows the second-by-second stock prices of Citigroup Inc on May 1, 2007, indicating that the log prices of the stock did not exhibit the pattern of a diffusion process or a diffusion process with additive noise. Rather, these prices seem like samples from a rounded diffusion.

In this article, we focus on the extreme case where there is pure rounding. We explore the commonly used volatility estimator, the realized volatility (RV) and how it can be bias-corrected to yield consistent volatility estimates. RV goes back to the path breaking work of Andersen and Bollerslev (1997), Andersen et al. (2001, 2003), Barndorff-Nielsen and Shephard (2002), Jacod and Protter (1998), among others.

We consider a security price process S, whose logarithm $X = \log S$ follows

$$dX_t = \mu_t dt + \sigma_t dW_t. \tag{1}$$

In other words, S is the solution to the following stochastic differential equation:

$$dS_t = (\mu_t + \frac{1}{2}\sigma_t^2) S_t dt + \sigma_t S_t dW_t, \;\; t \in [0,1] \tag{2}$$

where W_t is a standard Brownian motion. We assume that μ_t and σ_t are continuous random processes satisfying the regularity conditions specified in Section 1.

It is a common practice in finance to use the sum of frequently sampled squared returns, the RV, to estimate the integrated volatility $\int_0^1 \sigma_t^2 dt$. However, empirical studies have shown that because of market microstructure noise, RV can be severely biased when prices are sampled at high frequencies, whereas sampling sparsely yields more reasonable estimates (see, for example, the signature plots introduced by Andersen et al. (2000)). In this study, we investigate the case in which the contamination caused by market microstructure results solely from rounding errors.

Let α_n be a sequence of positive numbers representing the accuracy of measurement when the price process is observed n times during a time period [0,1]. Suppose that at time i/n ($i = 0, \cdots n$), the value $k\alpha_n$ is observed when the true value $S_{i/n}$ is in $[k\alpha_n, (k+1)\alpha_n)$ with $k \in \mathbb{Z}$. For every real s, we denote by $s^{(\alpha_n)} = \alpha_n \lfloor s/\alpha_n \rfloor$ its rounded-off value at level α_n. Taking the Citigroup data as in Figure 1 for example, the rounding level is $\alpha_n = 0.01$. On the day shown, the k ranged from 296 to 317.

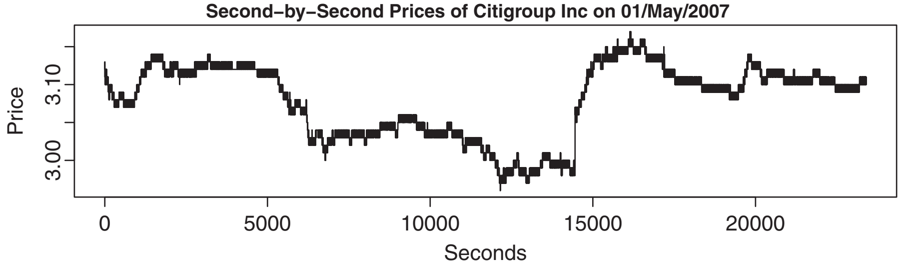

Figure 1 Second-by-second stock prices of Citigroup Inc on May 1, 2007. Rounding appears to be a main feature of the data.

We investigate the asymptotic behavior of the RV, as follows:

$$V^n = \sum_{i=1}^{n} (Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}})^2, \tag{3}$$

where $Y_{\frac{i}{n}} = \log(S_{i/n}^{(\alpha_n)})$, $i=0,\cdots,n$ represents the observed log prices. Our main results are presented for the case in which the rounding is *down* as previously described. This is primarily to facilitate presenting the proofs, where results of Delattre and Jacod (1997) for round-off errors are applied. Same or similar results also apply to rounding *up* or rounding *to the nearest multiple of* α_n (see Remark 5 and Section 2.3.2 for additional details).

Jacod (1996) and Delattre and Jacod (1997) previously studied volatility interference based on a rounded Itô diffusion; although their work inspired the current study, we seek in this article to spell out what ensues when rounding occurs on the original (e.g., the US dollar, euro, etc.) scale rather than the log scale. As we shall see later in this article, our findings indicate that this more realistic rounding yields a bias which requires a somewhat more complicated correction. For example, in the simple case that the volatility is constant, the bias is no longer a function of the volatility (see Remark 1).

We shall provide the limit of V^n, demonstrating that the RV can be problematic when rounding errors are present, and elucidating why "sparse sampling" could be a practical way to estimate volatility (however, sparsely sampling does not solve all the problems). We subsequently propose a bias-corrected estimator, and prove an associated central limit theorem. The simulation results demonstrate that the proposed bias-corrected estimator yields substantially enhanced statistical accuracy. Empirical studies show that the bias-correction can facilitate financial risk management. Our main bias correction applies to the case of "small rounding" as in Delattre and Jacod (1997) and Rosenbaum (2009). Such asymptotics are realistic in practice, cf. the findings for additive error in Zhang et al. (2011). Small rounding asymptotics has also been studied in Kolassa and McCullagh (1990), where it is shown to be related to additive error. We also discuss the effects on RV when the rounding is not "small".

The theoretical results are presented in Section 1; the simulation studies are presented in Section 2, and the empirical studies in Section 3; Section 4 concludes. The proofs are shown in the Appendix.

1 ASYMPTOTIC RESULTS

We assume that the latent security price process S_t follows (2), where σ_t is a nonrandom function of S_t, of class $\mathcal{C}^5$ on $[0,\infty)$ (In the Black–Scholes model, $\sigma_t \equiv \sigma$ is a constant). Assume further that μ_t is a continuous random process (in particular, it is locally bounded).

Let $\beta_n = \alpha_n\sqrt{n}$.

Theorem 1: *When $\alpha_n \to 0$ as $n \to \infty$ such that $\beta_n \to \beta \in [0,\infty)$, we have*

$$V^n \to_P \int_0^1 \sigma_t^2 dt + \frac{\beta^2}{6}\int_0^1 \frac{1}{S_t^2}dt - \frac{\beta^2}{\pi^2}\int_0^1 \frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)dt.$$

One sees from this result that the bias is always positive when $\beta \neq 0$, rapidly increasing as β grows. Also, the bias is smaller when the stock price $S_t, t \in [0,1]$ is larger. Figure 2 gives a visual representation of this. This result captures the empirical features that
a) subsampling helps (the same α value and a smaller n value yields a smaller β and correspondingly smaller bias); and
b) the rounding effect is less severe for more expensive stocks (i.e., higher S_t values correspond to smaller biases).

Theorem 1 shows that when $\beta_n \to 0$, one has the consistency of V^n. If β_n decays polynomially in n, we have the following central limit theorem.

Theorem 2: *When $\beta_n = O(n^{-\gamma})$ for some $\gamma > 0$, we have*

$$\sqrt{n}\left(V^n - \int_0^1 \sigma_t^2 dt - \frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt\right) \to_{\mathcal{L}-stably} \int_0^1 \sqrt{2}\sigma_t^2 dB_t,$$

where B is a standard Brownian motion independent of W.

In this case, a finite sample bias of $\frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt$ remains. The bias can be estimated and a bias-corrected estimator can be determined as follows.

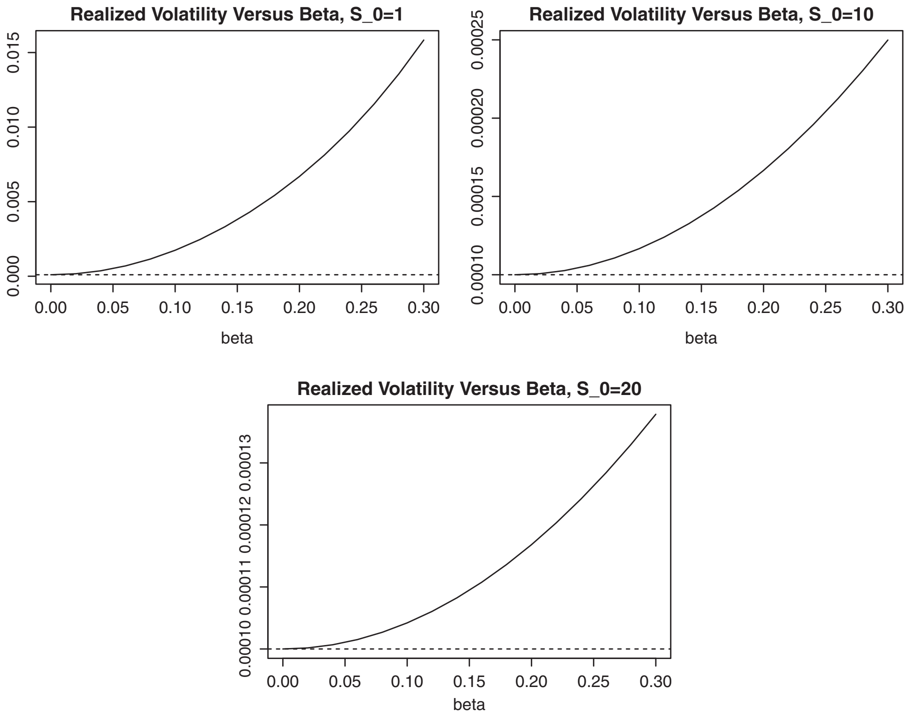

Figure 2 RV V^n versus β based on Theorem 1 and three simulated sample paths ($\mu_t \equiv 0$, $\sigma_t \equiv 0.01$) with starting prices $S_0 = 1$, $S_0 = 10$ and $S_0 = 20$. The dashed line represents the true integrated volatility, which is 0.0001; the solid curves represent the limits of the RV. The curve shapes indicate that the bias is increasing in β, and comparing the ranges of the y axes of the plots demonstrates that the bias is smaller when S_t, $t \in [0,1]$ is larger.

Theorem 3: *Assume that $\beta_n = O(n^{-\gamma})$ for some $\gamma > 0$, and let*

$$V_0^n := V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^{n} \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

Then as $n \to \infty$,

$$\sqrt{n}\left(V_0^n - \int_0^1 \sigma_t^2 dt\right) \to_{\mathcal{L}-stably} \int_0^1 \sqrt{2}\sigma_t^2 dB_t,$$

where B is a standard Brownian motion independent of W.

The simulation results in the subsequent section demonstrate that this bias-correction yields substantially improved estimates. The empirical studies further show that the bias correction can be quite helpful in risk analysis.

Remark 1: *In the case where $\sigma_t \equiv \sigma$, it is documented in section 4 of Delattre and Jacod (1997) that when rounding is implemented on the log scale, the bias is a function of $\sigma^2 = \int_0^1 \sigma_t^2 dt$. We see from the above results that in the presence of this more realistic type of rounding, the bias is no longer a function of the targeted integrated volatility even when $\sigma_t \equiv \sigma$. Therefore this requires somewhat more complicated bias correction.*[1]

Remark 2: *The condition of small rounding is necessary for the asymptotic results above. In practice, we apply these asymptotic results via expansion—we observe only one α_n and one n for a particular price process in a specified time period. When small rounding is relevant, we can make a correction as in Theorem 3, yielding a superior estimator. In practice, we usually cannot change α_n. If the β_n is too big due to high sampling frequency, we can analyze a subsample with a moderate frequency to establish a situation with small β_n. (Ref. simulation studies for additional detail.)*

Remark 3: *The condition that the random process σ_t is a nonrandom function of S_t is assumed so that the framework of Delattre and Jacod (1997) can be applied. In Sections 2 and 3, we see in simulation and empirical studies that even when this condition is not necessarily satisfied, the proposed bias correction can still be very helpful. We conjecture that similar results hold also in stochastic volatility settings.*[2]

When the small rounding condition is not satisfied, the RV would blow up as the sampling frequency becomes larger. Theorem 4 illustrates the asymptotic result of a simple case where $\sigma_t \equiv \sigma$. In this case, simple bias correction is insufficient. A correction after subsampling may help.

Theorem 4: *Let the accuracy of measurement $\alpha_n \equiv \alpha$ be independent of the number of observations n. Consider the case when $\sigma_t \equiv \sigma$ for $t \in [0,1]$. Redefine $S_{i/n}^{(\alpha)} = \alpha$ if $S_{i/n}^{(\alpha)} = 0$. As $n \to \infty$,*

$$\frac{1}{\sqrt{n}} V^n \to_P \frac{1}{\sigma}\sqrt{\frac{2}{\pi}} \sum_{k=1}^{\infty} L_1^{\log((k+1)\alpha)} \left(\log \frac{k+1}{k}\right)^2,$$

where L_t^a is the local time at the level a of the continuous semimartingale $X_t = \log S_t$ (see Revuz and Yor (1999), page 222).

Remark 4: *Redefining $S_{i/n}^{(\alpha)} = \alpha$ if $S_{i/n}^{(\alpha)} = 0$ rules out the possibility of yielding a logarithm of zero for log prices. In practice, this simply means that the security price does not go below the smallest rounding grid (1 cent if $\alpha = 0.01$) during the specified time period.*

[1] We emphasize that our derivation builds on the general results of Delattre and Jacod (1997).

[2] A formal extension to this more general case can use a simple parametric approximation to the process, perhaps via the contiguity arguments in Mykland and Zhang (2011).

Remark 5: *When rounding is not down, but rather to the nearest multiple of α_n, the results of Theorems 1–3 remain the same, but a small adjustment must be made to the limit of Theorem 4: the local times will be at levels* $\log((k+\frac{1}{2})\alpha)$ *instead of* $\log((k+1)\alpha)$.

2 SIMULATION STUDIES

2.1 Moderate Sampling Frequencies

Consider first the simplest case that $\sigma_t \equiv \sigma$ for $t \in [0,1]$. Denote by V^n_CI and $V_0^n\_CI$ the nominal 95% confidence interval (CI) based on V^n and V_0^n, respectively.

The naïve CI based on V^n relies on the classical theory for RV, which indicates the following when there is no observation error:

$$\sqrt{n}[V^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

The resulting nominal 95% CI is as follows:

$$V^n\_CI = \left[V^n - 1.96 * \sqrt{2(V^n)^2/n},\ V^n + 1.96 * \sqrt{2(V^n)^2/n} \right].$$

Our findings in the previous section indicate that the RV is no longer reliable when rounding errors are present. We proposed the following simple bias-corrected estimator that should function when $\alpha_n\sqrt{n}$ is reasonably small:

$$V_0^n = V^n - \frac{\alpha_n^2}{6} \sum_{i=1}^{n} \frac{1}{(S_{i/n}^{(\alpha_n)})^2}.$$

By Theorem 3,

$$\sqrt{n}[V_0^n - \sigma^2] \to_{\mathcal{L}} N(0, 2\sigma^4).$$

The adjusted nominal 95% CI is as follows:

$$V_0^n\_CI = \left[V_0^n - 1.96 * \sqrt{2(V_0^n)^2/n},\ V_0^n + 1.96 * \sqrt{2(V_0^n)^2/n} \right].$$

To examine the performance of the volatility estimators V^n and V_0^n, we perform the following simulation study.

We simulate sample paths from (2) with $\mu = 0$, $\sigma = 0.01$. We examine two price levels, with starting prices of $S_0 = \$10$ and $S_0 = \$50$, respectively. At each price level, 10,000 simulations were conducted for a one-day period. We use a fixed rounding level of $\alpha_n \equiv \alpha = 0.01$, to mimic the financial market, in which stock prices are often rounded to the cent.

Table 1 Performance of the nominal 95% CIs based on V^n and V_0^n for stocks priced at approximately \$10 ($S_0 = \$10$)

samp. freq.	samp. intv.	‘β_n’		V^n_CI	V_0^n_CI
			f:	94.29%	89.57%
78	5 min	0.088	l:	$7.02*10^{-5}$	$6.20*10^{-5}$
			b:	$1.18*10^{-5}$	$-1.21*10^{-6}$
			f:	78.59%	87.78%
130	3 min	0.114	l:	$5.87*10^{-5}$	$4.81*10^{-5}$
			b:	$2.06*10^{-5}$	$-1.02*10^{-6}$
			f:	31.29%	85.08%
195	2 min	0.140	l:	$5.23*10^{-5}$	$3.94*10^{-5}$
			b:	$3.18*10^{-5}$	$-6.57*10^{-7}$
			f:	0	74.75%
390	1 min	0.197	l:	$4.61*10^{-5}$	$2.79*10^{-5}$
			b:	$6.4*10^{-5}$	$-6.71*10^{-7}$
			f:	0	46.91%
780	30 sec	0.279	l:	$4.44*10^{-5}$	$1.85*10^{-5}$
			b:	$1.23*10^{-4}$	$-6.54*10^{-6}$

“f”: actual coverage frequency of the CIs; “l”: average CI length; “b”: finite sample bias.

Tables 1 and 2 show the simulation results. The first columns show the sample frequencies (`samp. freq.`), the second columns show the corresponding sample intervals (`samp. intv.`), and the third columns show the pre-limiting $\beta_n = \alpha\sqrt{n}$, demonstrating how the proposed small rounding asymptotic theory functions at a finite sample size and fixed rounding level. The final two columns display three items each. The notation “f” denotes the “actual coverage frequency,” which is used to record the frequency at which the true parameter is covered by the CIs based on the corresponding volatility estimators V^n and V_0^n; “l” denotes the “average length of the CI,” which indicates the CI width; and “b” denotes the “finite sample bias,” which indicates how much and in which direction the estimators are biased.

Comparing V^n with V_0^n indicates that when the sample frequency is relatively low (e.g., a 5-min sampling interval for stocks priced at approximately \$10, or 1–5 min for stocks priced at approximately \$50), both V^n and V_0^n perform well because their actual coverage frequency is near the nominal rate of 95%. This is consistent with the empirical evidence that subsampling is beneficial. But since the convergence rate is the square root of n, the CIs are wide when n is small. When the sample frequency increases slightly (e.g., 3 min - 1 min for stocks priced at approximately \$10 or 30 sec - 20 sec for stocks priced at approximately \$50), the problems with the RV become apparent, the coverage frequency decreases from approximately 95% to some much lower rates (or even zero in the former case), whereas the V_0^n_CI continues to exhibit a large coverage frequency. The biases demonstrate that the RV goes to some values much larger than the true value,

Table 2 Performance of the nominal 95% CIs based on V^n and V_0^n for stocks priced at approximately \$50 ($S_0 = \$50$)

samp. freq.	samp. intv.	'β' $\alpha\sqrt{n}$		V^n_CI	V_0^n_CI
			f:	92.89%	92.48%
78	5 min	0.088	l:	$6.23*10^{-5}$	$6.19*10^{-5}$
			b:	$-7.51*10^{-7}$	$-1.27*10^{-6}$
			f:	94.01%	93.49%
130	3 min	0.114	l:	$4.86*10^{-5}$	$4.82*10^{-5}$
			b:	$9.39*10^{-8}$	$-7.73*10^{-7}$
			f:	94.83%	93.86%
195	2 min	0.140	l:	$3.99*10^{-5}$	$3.94*10^{-5}$
			b:	$6.26*10^{-7}$	$-6.74*10^{-7}$
			f:	94.9%	93.81%
390	1 min	0.197	l:	$2.87*10^{-5}$	$2.80*10^{-5}$
			b:	$2.31*10^{-6}$	$-2.95*10^{-7}$
			f:	86.01%	93.45%
780	30 sec	0.279	l:	$2.08*10^{-5}$	$1.98*10^{-5}$
			b:	$5.08*10^{-6}$	$-1.23*10^{-7}$
			f:	60.83%	93.12%
1170	20 sec	0.342	l:	$1.74*10^{-5}$	$1.62*10^{-5}$
			b:	$7.66*10^{-6}$	$-1.37*10^{-7}$
			f:	0.27%	31.45%
2340	10 sec	0.484	l:	$1.32*10^{-5}$	$1.23*10^{-5}$
			b:	$1.55*10^{-5}$	$7.72*10^{-6}$

"f": actual coverage frequency of the CIs; "l": average CI length; "b": finite sample bias.

whereas the V_0^n remains close to the true parameter value. Hence, V_0^n substantially outperforms the uncorrected RV V^n.

At extremely high frequencies (less than 30 sec for \$10 stocks or less than 10 sec for \$50 stocks), the bias-corrected volatility estimator V_0^n demonstrates decreased performance levels, although its bias remains substantially smaller compared with the RV). This is expected, because the bias-corrected estimator is built on asymptotic theory, which requires the condition $\alpha_n\sqrt{n} \to 0$, which is hypothetical, since in practice one is faced with a fixed data set and a fixed tick size. If the sample frequency were to continue to increase at a fixed rounding level, the proposed bias correction would eventually fail. The failure at extremely high frequency is expected among other RV-based volatility estimators as well and is a direct consequence of Theorem 4 (Theorem 2 in Li and Mykland (2007) provides the result for the two scales realized volatility of Zhang et al. (2005)). The above simulation shows that for a given price level and rounding level, the proposed bias correction method is effective when the sample frequency is not excessively high.

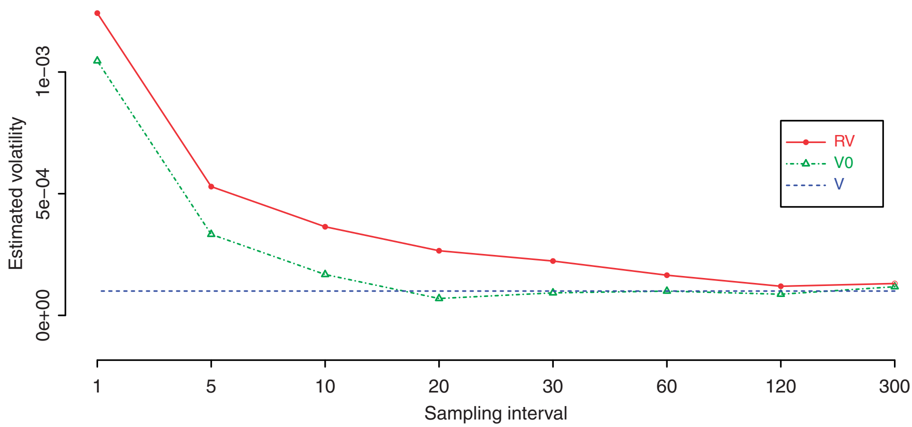

Figure 3 V^n (RV) and V_0^n ($V0$) at various sampling frequencies when the rounding level is fixed at 0.01 for a sample path. The dotted horizontal line is the true integrated volatility (V). The sampling frequencies vary from 1 observation per second (left) to 1 observation per 300 seconds (right). The price level is approximately \$10 in this simulation.

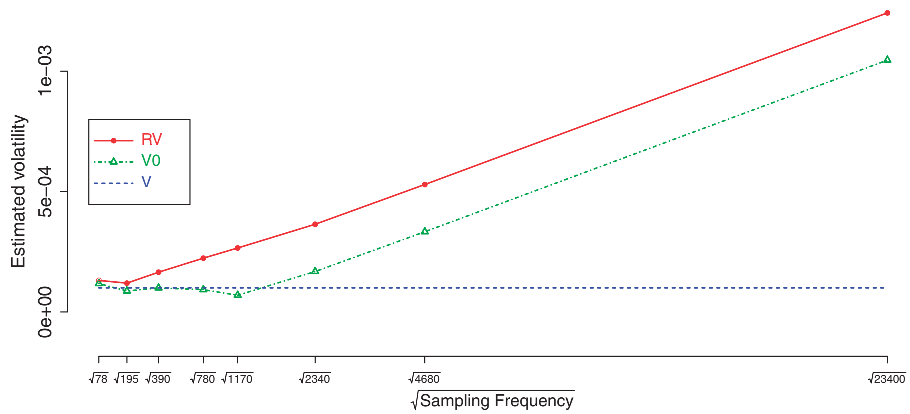

Figure 4 V^n (RV) and V_0^n ($V0$) versus the square root of the sampling frequencies when the rounding level is fixed at 0.01 for a sample path. The dotted horizontal line is the true integrated volatility (V). The sampling frequencies vary from 1 observation per second (right) to 1 observation per 300 seconds (left). The price level is approximately \$10 in this simulation.

2.2 Large Sampling Frequencies at a Fixed Rounding Level

At a fixed rounding level, when n is excessively large, the conditions of Theorems 1–3 are no longer met and the feature described in Theorem 4 appears (Figures 3 and 4).

Figures 3 and 4 demonstrate that as the sampling frequency increases (sampling interval decreases), both the RV and the proposed bias corrected estimator V_0^n will

Table 3 Performance of V^n and V_0^n when volatility is not a function of price

	1st Quartile	Median	3rd Quartile	Mean	Root Mean Squares
$V^n - V$	$1.28*10^{-4}$	$2.05*10^{-4}$	$3.15*10^{-4}$	$2.47*10^{-4}$	$3.02*10^{-4}$
$V_0^n - V$	$-4.36*10^{-5}$	$-1.02*10^{-6}$	$1.74*10^{-5}$	$-1.85*10^{-5}$	$6.98*10^{-5}$

The estimation errors are summarized by their 1st quartile, median, 3rd quartile, mean, and root mean squares.

rapidly increase as described in Theorem 4. Figure 4 most clearly exhibits the rate of divergence. However, the V_0^n demonstrates a clear advantage over the RV at a large range of moderate-sized sampling intervals (20s - 2 min in the case illustrated in Figure 4).

2.3 When Conditions Deviate From the Requirements

2.3.1 When volatility is not a function of price. The theoretical results are established under conditions specified in Section 1; it is worth investigating how the bias correction performs if the required conditions are not met. Therefore, we conduct simulations based on a stochastic model in which the volatility process evolves by itself and is *not a function of the price process*. The Heston model (Heston 1993) was adopted to determine the log price:

$$dX_t = (\mu - \nu_t/2)dt + \sigma_t dB_t$$

$$d\nu_t = \kappa(\eta - \nu_t)dt + \gamma \nu_t^{1/2} dW_t$$

where $\nu_t = \sigma_t^2$, B and W are standard Brownian motions with $E(dB_t dW_t) = \rho dt$, and the parameters $\mu, \eta, \kappa, \gamma, \rho$, and the starting log-price X_0 are set at 0.05/252, 0.1/252, 5/252, 0.5/252, −0.5, and log(9), respectively. Aït-Sahalia and Kimmel (2007) and Aït-Sahalia et al. (2013) were referenced when selecting these parameter values. We used a moderate leverage effect parameter $\rho = -0.5$ to represent an individual stock. We simulate 10,000 days and obtained pairs of the latent observations X_{t_i}, σ_{t_i} for $t_0 = 0, t_1 = \frac{1}{390}, \cdots, t_n = 1$ for each day (one observation per minute, $n = 390$). We compute the integrated volatility $V = n^{-1} \sum_{i=1}^{n} \sigma_{t_i}^2$ and use this value as the reference measure. The observed log prices are $\log(\exp(X_{t_i})^{(\alpha)})$ where $\alpha = 0.01$ (rounded to cent). We compute the RV V^n and the proposed bias-corrected estimator V_0^n; and summarize their performance in Table 3.

The results indicate that although this model fails to meet the conditions for the theoretical results, the estimator V_0^n demonstrates a clear advantage.

2.3.2 When rounding to the nearest multiple of α. As mentioned in Remark 5, rounding down and rounding to the nearest multiple of α should yield

Table 4 Performance of V^n and V_0^n when rounding to the nearest multiple of α

	1st Quartile	Median	3rd Quartile	Mean	Root Mean Squares
V^n-V	$1.28*10^{-4}$	$2.01*10^{-4}$	$3.13*10^{-4}$	$2.45*10^{-4}$	$3.00*10^{-4}$
V_0^n-V	$-4.54*10^{-5}$	$-1.14*10^{-6}$	$1.79*10^{-5}$	$-1.89*10^{-5}$	$7.09*10^{-5}$

The estimation errors are summarized by their 1st quartile, median, 3rd quartile, mean, and root mean squares.

similar results. Table 4 presents the results, when comparing the performance of V_0^n and V^n based on the same simulated sample paths as the one studied in Table 3. The only difference is that the observations become $Y_{i/n}=\log[\frac{\exp(X_{t_{i/n}})}{\alpha}]\cdot\alpha$ with $\alpha=0.01$ $(i=1,\cdots,n)$, where $[\cdot]$ denotes rounding to the nearest integer. The results shown in Table 4 are similar to those in Table 3, as expected.

2.3.3 When jumps exist. We further investigate volatility estimation in the presence of jumps. We simulated the following model:

$$dX_t=(\mu-\nu_t/2)dt+\nu_t^{1/2}dB_t+J_t dN_t \tag{4}$$

$$d\nu_t=\kappa(\eta-\nu_t)dt+\gamma\nu_t^{1/2}dW_t, \tag{5}$$

where B_t and W_t are Brownian motions with correlation ρ, N_t is a Poisson process with intensity λ, and J_t denotes the jump size, which is assumed to follow an independent $N(0,\sigma_J^2)$. The prices are again rounded to cents: $Y_{i/n}=\log(\exp(X_{i/n})^{(\alpha)})$ $(i=1,\cdots,n)$.

One way to remove the impact of jumps in volatility estimation is using the truncated RV which is defined as follows (Aït-Sahalia and Jacod, 2012).[3]

$$V^{n,tr}=\sum_{i=1}^{n}(Y_{i/n}-Y_{(i-1)/n})^2 1_{\{|Y_{i/n}-Y_{(i-1)/n}|\le an^{-\varpi}\}} \tag{6}$$

for some $\varpi\in(0,1/2)$ and $a>0$.

We define

$$V_0^{n,tr}=V^{n,tr}-\frac{\alpha_n^2}{6}\sum_{i=1}^{n}\frac{1}{(\exp(Y_{i/n}))^2}$$

as the bias-corrected version of the truncated RV. The parameters $\eta=0.1$, $\gamma=0.5/252$, $\kappa=5/252$, $\rho=-0.5$, $\mu=0.05/252$, $\lambda=5$, $\sigma_J=0.015$, and

[3]See also Mancini (2001), Lee and Mykland (2008), and Jing et al. (2012). Bi- and multipower methods (Barndorff-Nielsen and Shephard, 2004, 2006) may also work, but we have not investigated this.

Table 5 Performance of $V^{n,tr}$ and $V_0^{n,tr}$ when jumps exist

	1st Quartile	Median	3rd Quartile	Mean	Root mean squares
$V^{n,tr}-V$	$1.47*10^{-4}$	$2.24*10^{-4}$	$3.34*10^{-4}$	$2.59*10^{-4}$	$3.35*10^{-4}$
$V_0^{n,tr}-V$	$-8.46*10^{-5}$	$-2.96*10^{-5}$	$6.03*10^{-5}$	$-7.12*10^{-5}$	$2.29*10^{-4}$

The estimation errors are summarized by their 1st quartile, median, 3rd quartile, mean and root mean squares.

$X_0=\log(9)$ were used, and the truncation level was set at $an^{-\varpi}=4\sqrt{\alpha}n^{-1/2}$ (ref. Aït-Sahalia and Jacod (2012), Aït-Sahalia et al. (2013)). The results are summarized in Table 5, which indicates that when jumps exist, the proposed bias correction method plays a significant role in reducing the bias of the truncated RV.

Remark 6: *We notice that there is some deterioration in small sample behavior relative to the no jump-no truncation case. Naturally, the truncation leads to both higher bias and higher variance in small samples. The issue may relate to whether jumps get over-detected in small samples (Bajgrowicz et al., 2013), but also to the fact that one loses intervals that have continuous evolution (whether or not they have jumps). This latter problem has been discussed, with possible solutions, in Lee (2005) and Lee and Hannig (2010), and is beyond the scope of this article.*

3 EMPIRICAL STUDY

To further illustrate the effectiveness of the proposed bias correction method, we conduct an empirical analysis of Citigroup Inc. (NYSE:C), CBS Corporation (NYSE:CBS), Dell Inc. (NYSE:DELL), Host Hotels and Resorts Inc. (NYSE:HST), and KeyCorp (NYSE:KEY) stock data from 2009. We collected the stock prices every minute (390 observations per day), computing the V^n and V_0^n values for each day. Based on the estimated volatilities, and the assumption that the return on each day is normally distributed, exhibiting approximately zero mean and variance as estimated (as is commonly assumed in risk management[4]), we computed the 5% value at risk (VaR) for each day, counting the total number of days that the VaR was violated. Table 6 lists the VaR violations that occurred among the 252 days considered.

Because we considered the 5% VaRs, the expected rate of violation was 5%. The examined stocks based on the V_0^n demonstrated rates closer to the expected rate than did those based on V^n. The V^n tends to dramatically overestimate the daily volatilities, causing over-cautious VaRs.

[4]See, for example, Christoffersen (2011). We used the estimated volatility of the *same day* for computing VaR of that day as our main purpose here is to examine the accuracy of the volatility estimators instead of to predict VaR.

Table 6 5% VaR violation rate based on the minute-by-minute stock prices of C, CBS, DELL, HST, and KEY stocks in 2009

	C	CBS	DELL	HST	KEY
V^n	1.59%	2.78%	3.57%	3.57%	2.38%
V_0^n	2.78%	3.17%	4.76%	3.97%	3.97%

4 CONCLUSIONS AND DISCUSSION

We have explored the estimation of the integrated volatility when rounding is the primary source of market microstructure noise. We established asymptotic results for the RV based on "small rounding" conditions. We proposed a bias-corrected estimator for which consistency and central limit theorems were established. Results were also presented for the case when "small rounding" conditions were not satisfied. The effectiveness and practicailty of the proposed bias correction method was demonstrated in both simulation and empirical studies.

Note that while we work with observations on a time interval $[0,1]$, results for the more general case of time interval $[0,T]$ are obtained by rescaling. The case of unequal observation times can be studied based on the methods of Jacod and Protter (1998) and Mykland and Zhang (2006).

APPENDIX

A.1 Preparation

We assume without loss of generality (see Section A.4 for further justification) that $\mu_t = 0$, in which case

$$d\log S_t = \sigma_t dW_t; \tag{A.1}$$

and that there exist nonrandom constants L_σ, $U_\sigma \in (0,\infty)$, such that

$$L_\sigma \leq \sigma_t \leq U_\sigma \quad \text{for } t \in [0,1].$$

More Notation:

$$A_m := \left\{ \omega \in \Omega : S_t(\omega)_{t\in[0,1]} \in [\frac{1}{m}, m] \right\};$$

$$B_n := \left\{ \omega \in \Omega : \max_{1\leq i\leq n} \sqrt{n} \left| \frac{S_{i/n}}{S_{(i-1)/n}} - 1 \right| \leq 2\log n \right\};$$

$$Y_{i,n} := \sqrt{n} \left(S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)} \right);$$

$$U(n,\phi) := \frac{1}{n} \sum_{i=1}^{n} \phi\left(S_{(i-1)/n}^{(\alpha_n)}, Y_{i,n} \right) \text{ for } \phi : \mathbb{R}^2 \to \mathbb{R}; \tag{A.2}$$

$$h(\cdot):\ \text{density of the standard normal law;}$$
$$h_s(\cdot):\ \text{density of the normal law } N(0,s^2).$$

Lemma 1: $P(B_n)\to 1$ *as* $n\to\infty$.

Proof. By (A.1),

$$S_{i/n}/S_{(i-1)/n}=\exp\left(\int_{(i-1)/n}^{i/n}\sigma_s dW_s\right).$$

Note that for any $i=1,2,\cdots n$,

$$\begin{aligned}
&E\left(\exp\left(\sqrt{n}\int_{(i-1)/n}^{i/n}\sigma_s dW_s\right)\right)\\
\leq &E\left(\exp\left(\sqrt{n}\int_{(i-1)/n}^{i/n}\sigma_s dW_s-\frac{1}{2}n\int_{(i-1)/n}^{i/n}\sigma_s^2 ds+\frac{1}{2}U_\sigma^2\right)\right)\\
= &\exp\left(\frac{1}{2}U_\sigma^2\right).
\end{aligned}$$

Hence for any $a>0$

$$\begin{aligned}
&P\left(\int_{(i-1)/n}^{i/n}\sigma_s dW_s>a\right)\\
= &P\left(\exp\left(\sqrt{n}\int_{(i-1)/n}^{i/n}\sigma_s dW_s\right)>\exp\left(\sqrt{n}a\right)\right)\\
\leq &\frac{\exp\left(\frac{1}{2}U_\sigma^2\right)}{\exp\left(\sqrt{n}a\right)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P\left(\max_{1\leq i\leq n}\left(\sqrt{n}\left(\frac{S_{i/n}}{S_{(i-1)/n}}-1\right)\right)>2\log n\right)\\
= &P\left(\max_{1\leq i\leq n}\left(\frac{S_{i/n}}{S_{(i-1)/n}}\right)>\frac{2\log n}{\sqrt{n}}+1\right)\\
= &P\left(\max_{1\leq i\leq n}\left(\int_{(i-1)/n}^{i/n}\sigma_s dW_s\right)>\log\left(\frac{2\log n}{\sqrt{n}}+1\right)\right)
\end{aligned}$$

$$\leq n\frac{\exp\left(\frac{1}{2}U_\sigma^2\right)}{\exp\left(\sqrt{n}(\log(\frac{2\log n}{\sqrt{n}}+1))\right)}$$

$$\to 0 \text{ as } n\to\infty.$$

A parallel argument gives

$$P\left(\max_{1\leq i\leq n}\left(\sqrt{n}\left(1-\frac{S_{i/n}}{S_{(i-1)/n}}\right)\right)>2\log n\right)\to 0 \text{ as } n\to\infty,$$

hence the conclusion. ■

Lemma 2: *If $\sqrt{n}\alpha_n\to\beta\in[0,\infty)$, then for any m, there exist N large and $c_m\in(0,\frac{1}{m}]$ such that for all $n\geq N$, $i=0,1,2,\cdots,n$,*

$$S_{i/n}^{(\alpha_n)}\geq c_m \text{ on } A_m.$$

Proof.

$$\forall i=0,1,2,\cdots,n,\ S_{i/n}^{(\alpha_n)}\geq S_{i/n}-\alpha_n;$$

and

$$S_{i/n}\geq\frac{1}{m} \text{ on } A_m, \text{ and } \alpha_n\to 0 \text{ as } n\to\infty,$$

hence the conclusion. ■

Lemma 3: *Suppose that $\beta_n=\sqrt{n}\alpha_n\to\beta\in[0,\infty)$, then for any fixed $m>0$,*

$$\sup_{\omega\in A_m\bigcap B_n}\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}=O\left(\frac{\log n}{\sqrt{n}}\right).$$

Proof. On $A_m\bigcap B_n$,

$$|Y_{i,n}|=\sqrt{n}|S_{i/n}^{(\alpha_n)}-S_{(i-1)/n}^{(\alpha_n)}|\leq\sqrt{n}(|S_{i/n}-S_{(i-1)/n}|+2\alpha_n)\leq 2m\log n+2\beta_n.$$

By Lemma 2, one can find a $c_m\in(0,\frac{1}{m}]$ such that for large n, on $A_m\bigcap B_n$,

$$\frac{|Y_{i,n}|}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}\leq\frac{2m\log n+2\beta_n}{\sqrt{n}c_m}.$$

Since $\beta_n\to\beta<\infty$, the above inequality implies that for any fixed m, $\sup_{\omega\in A_m\bigcap B_n}\frac{Y_{i,n}}{\sqrt{n}S_{(i-1)/n}^{(\alpha_n)}}$ is $O\left(\frac{\log n}{\sqrt{n}}\right)$. ■

Lemma 4: *Let $\beta > 0$, then for all $\sigma, x > 0$,*

$$\int_0^1 \int h(y) \left(\frac{\beta \lfloor u + y\sigma x/\beta \rfloor}{x} \right)^2 dydu = \sigma^2 + \frac{1}{x^2} \left(\frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left(-2\pi^2 k^2 \frac{\sigma^2 x^2}{\beta^2} \right) \right).$$

Proof.

$$\begin{aligned}
&\int_0^1 \int h(y) \left(\frac{\beta \lfloor u + y\sigma x/\beta \rfloor}{x} \right)^2 dydu \\
=&E\left(\frac{\beta \lfloor U + Y\sigma x/\beta \rfloor}{x} \right)^2, \, U \sim unif[0,1], \, Y \sim N(0,1) \\
=&\frac{\beta^2}{x^2} E(\lfloor U + Y\sigma x/\beta \rfloor)^2 \\
=&\frac{\beta^2}{x^2} E(\lfloor U + Z \rfloor)^2, \, Z \sim N\left(0, \frac{\sigma^2 x^2}{\beta^2}\right) \\
=&\frac{\beta^2}{x^2} E\Big(E(\lfloor U + Z \rfloor^2 | Z) \Big) \\
=&\frac{\beta^2}{x^2} E\Big((Z - \{Z\})^2 (1 - \{Z\}) + (Z + 1 - \{Z\})^2 \{Z\} \Big) \\
=&\frac{\beta^2}{x^2} \Big(EZ^2 + E(\{Z\}(1 - \{Z\})) \Big) \\
=&\sigma^2 + \frac{1}{x^2} \left(\frac{\beta^2}{6} - \frac{\beta^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} \exp\left(-2\pi^2 k^2 \frac{\sigma^2 x^2}{\beta^2} \right) \right),
\end{aligned}$$

where $\{z\} = z - \lfloor z \rfloor$ is the fractional part of z.

The last equality above is proved by using the Fourier expansion for function $f(z) = \{z\} - \{z\}^2$. ■

A.2 Proof of Theorem 1

Recall that V^n is defined in (3). For large n,

$$\begin{aligned}
&V^n I_{A_m \cap B_n} \\
=&\sum_{i=1}^{n} (\log S_{i/n}^{(\alpha_n)} - \log S_{(i-1)/n}^{(\alpha_n)})^2 I_{A_m \cap B_n} \\
=&\frac{1}{n} \sum_{i=1}^{n} \left[\sqrt{n} \log\left(\frac{S_{i/n}^{(\alpha_n)} - S_{(i-1)/n}^{(\alpha_n)}}{S_{(i-1)/n}^{(\alpha_n)}} + 1 \right) \right]^2 I_{A_m \cap B_n}
\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{n}\log\left(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}+1\right)\right]^2 I_{A_m\cap B_n} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{n}\left(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}-\frac{1}{2}\left(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}\right)^2+\frac{1}{3}\theta^3\right)\right]^2 I_{A_m\cap B_n},
\end{aligned}
$$

$$
\text{for } \theta\in\left(0,\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}\right).
$$

By Lemma 2, one can find

$$
c_m\in(0,\frac{1}{m}] \text{ such that for large } n,\ S^{(\alpha_n)}_{i/n}\geq c_m \text{ for all } i=0,1,2,\cdots,n. \tag{A.4}
$$

Define

$$
\phi_{c_m}(x,y)=\begin{cases}\left(\dfrac{y}{x}\right)^2, & \text{when } x\geq c_m;\\[2mm] \left(\dfrac{3}{{c_m}^4}x^2-\dfrac{8}{{c_m}^3}x+\dfrac{6}{{c_m}^2}\right)y^2, & \text{when } x<c_m.\end{cases} \tag{A.5}
$$

Note in particular that ϕ_{c_m} is a function satisfying Hypothesis L_r in Delattre and Jacod (1997) with $r=2$.

For n large enough, by Lemmas 2 and 3, (A.3) can be rewritten as

$$
\begin{aligned}
V^n I_{A_m\cap B_n} &\leq \frac{1}{n}\sum_{i=1}^{n}\phi_{c_m}(S^{(\alpha_n)}_{(i-1)/n},Y_{i,n})I_{A_m\cap B_n}+O\left(\frac{(\log n)^3}{n^{1/2}}\right)I_{A_m\cap B_n}\\
&= U(n,\phi_{c_m})I_{A_m\cap B_n}+O\left(\frac{(\log n)^3}{n^{1/2}}\right)I_{A_m\cap B_n},
\end{aligned}
$$

where $U(\cdot,\cdot)$ is defined in (A.2).

Furthermore,

$$
\begin{aligned}
V^n I_{A_m} &= V^n I_{A_m\cap B_n}+V^n I_{A_m\cap B_n^c}\\
&\leq U(n,\phi_{c_m})I_{A_m}+(V^n-U(n,\phi_{c_m}))I_{A_m\cap B_n^c}+O\left(\frac{(\log n)^3}{n^{1/2}}\right)I_{A_m\cap B_n}\\
&= U(n,\phi_{c_m})I_{A_m}+o_p(1) \quad \text{(by Lemma 1).}
\end{aligned}
$$

By Theorem 3.1 of Delattre and Jacod (1997),

$$
U(n,\phi_{c_m})\to_P\begin{cases}\displaystyle\int_0^1\int_0^1\int h(y)\phi_{c_m}(S_t,\beta\lfloor u+y\sigma_t S_t/\beta\rfloor)dydudt, & \text{if } \beta>0;\\[2mm] \displaystyle\int_0^1\int h(y)\phi_{c_m}(S_t,y\sigma_t S_t)dydt, & \text{if } \beta=0.\end{cases}
$$

Note that since $c_m \le 1/m$, we have

$$\phi_{c_m}(S^{(\alpha_n)}_{(i-1)/n}, Y) = \left(\frac{Y}{S^{(\alpha_n)}_{(i-1)/n}}\right)^2 I_{A_m} + \phi_{c_m}(S^{(\alpha_n)}_{(i-1)/n}, Y) I_{A_m^c}.$$

Lemma 4 gives, when $\beta > 0$,

$$U(n, \phi_{c_m}) I_{A_m} \to_P \int_0^1 \frac{1}{S_t^2}\left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)\right) dt I_{A_m}.$$

It is easy to check that the above convergence is also true when $\beta = 0$. Therefore, for $\beta \in [0, \infty)$,

$$\begin{aligned} & V^n I_{A_m} \\ = & U(n, \phi_{c_m}) I_{A_m} + o_p(1) \\ \to_P & \int_0^1 \frac{1}{S_t^2}\left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)\right) dt I_{A_m}. \end{aligned}$$

That is to say, for any $\delta > 0$, $\epsilon > 0$, there exists N, such that for all $n > N$,

$$P\left(\left| V^n I_{A_m} - \int_0^1 \frac{1}{S_t^2}\left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)\right) dt I_{A_m}\right| > \delta\right) < \epsilon.$$

On the other hand, since $A_m \nearrow \Omega$ as $m \to \infty$, there exists M large, such that

$$P(A_M^c) < \epsilon.$$

Therefore, for $n > N$,

$$\begin{aligned} & P\left(\left| V^n - \int_0^1 \frac{1}{S_t^2}\left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)\right) dt\right| > \delta\right) \\ \le & P(A_M^c) + \\ & P\left(\left| V^n I_{A_M} - \int_0^1 \frac{1}{S_t^2}\left(\sigma_t^2 S_t^2 + \frac{\beta^2}{6} - \frac{\beta^2}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2 \frac{\sigma_t^2 S_t^2}{\beta^2}\right)\right) dt I_{A_M}\right| > \delta\right) \\ < & 2\epsilon. \end{aligned}$$

This proves Theorem 1.

A.3 Proof of Theorem 2 and Theorem 3

By (A.3), for large n,

$$\begin{aligned}&\sqrt{n}V^n I_{A_m\cap B_n}\\ =&\sqrt{n}\frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{n}\left(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}-\frac{1}{2}\left(\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}\right)^2+\frac{1}{3}\theta^3\right)\right]^2 I_{A_m\cap B_n},\end{aligned} \tag{A.6}$$

$$\text{for } \theta\in\left(0,\frac{Y_{i,n}}{\sqrt{n}S^{(\alpha_n)}_{(i-1)/n}}\right).$$

Using the $c_m\in(0,\frac{1}{m}]$ as in (A.4), we define

$$\psi_{c_m}(x,y)=\begin{cases}\left(\frac{y}{x}\right)^3, \text{ when } x\geq c_m;\\ \left(\frac{4}{{c_m}^3}-\frac{3x}{{c_m}^4}\right)y^3, \text{ when } x<c_m.\end{cases} \tag{A.7}$$

(A.6) can be further written as

$$\begin{aligned}&\sqrt{n}V^n I_{A_m\cap B_n}\\ \leq&\sqrt{n}U(n,\phi_{c_m})I_{A_m\cap B_n}-U(n,\psi_{c_m})I_{A_m\cap B_n}+O\left(\frac{(\log n)^4}{n^{1/2}}\right)I_{A_m\cap B_n};\end{aligned}$$

and

$$\begin{aligned}&\sqrt{n}V^n I_{A_m}\\ =&\sqrt{n}V^n I_{A_m\cap B_n}+\sqrt{n}V^n I_{A_m\cap B_n^c}\\ \leq&(\sqrt{n}U(n,\phi_{c_m})-U(n,\psi_{c_m}))I_{A_m}\\ &+(\sqrt{n}V^n-\sqrt{n}U(n,\phi_{c_m})+U(n,\psi_{c_m}))I_{A_m\cap B_n^c}+O\left(\frac{(\log n)^4}{n^{1/2}}\right)I_{A_m\cap B_n}\\ =&\sqrt{n}U(n,\phi_{c_m})I_{A_m}-U(n,\psi_{c_m})I_{A_m}+o_p(1),\end{aligned}$$

where ϕ_{c_m} is defined in (A.5), ψ_{c_m} in (A.7) and $U(\cdot,\cdot)$ in (A.2), and we have used Lemma 3 in the above.

Note that $\psi_{c_m}(S_t,\sigma_t S_t y)$ is an odd function of y, and $\beta=0$; by Theorem 3.1 of Delattre and Jacod (1997),

$$U(n,\psi_{c_m})\to_P\int_0^1\int h(y)\psi_{c_m}(S_t,\sigma_t S_t y)dydt=0.$$

Therefore,

$$U(n,\psi_{c_m})I_{A_m}\to_P 0.$$

As a consequence,

$$\sqrt{n}V^n I_{A_m}=\sqrt{n}U(n,\phi_{c_m})I_{A_m}+o_p(1). \tag{A.8}$$

Also by Corollary 3.3 of Delattre and Jacod (1997), since $\phi_{c_m}(x,y)$ is even in y,

$$\begin{aligned}&\sqrt{n}[U(n,\phi_{c_m})-\int_0^1\Gamma\phi_{c_m}(S_t,\beta_n)dt]\\ &\to_{\text{stably in law}}\int_0^1\Delta(\phi_{c_m},\phi_{c_m})(S_t,0)^{1/2}dB_s,\end{aligned} \tag{A.9}$$

where $B\perp\!\!\!\perp W$, and

$$\begin{aligned}&\Gamma\phi_{c_m}(S_t,\beta_n)\\ &=\int_0^1\int h(y)\phi_{c_m}(S_t,\beta_n\lfloor u+y\sigma_t S_t/\beta_n\rfloor)dydu\\ &=\int_0^1\int h(y)\left(\left(\frac{\beta_n\lfloor u+y\sigma_t S_t/\beta_n\rfloor}{S_t}\right)^2 I_{A_m}+\phi_{c_m}\left(S_t,\beta_n\lfloor u+y\sigma_t S_t/\beta_n\rfloor\right)I_{A_m^c}\right)dydu\\ &=\left(\sigma_t^2+\frac{\beta_n^2}{6}\frac{1}{S_t^2}-\frac{\beta_n^2}{\pi^2}\frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2k^2\frac{\sigma_t^2S_t^2}{\beta_n^2}\right)\right)I_{A_m}+\\ &\int_0^1\int h(y)\phi_{c_m}(S_t,\beta_n\lfloor u+y\sigma_t S_t/\beta_n\rfloor)dyduI_{A_m^c}\ \text{(by Lemma 4)};\end{aligned} \tag{A.10}$$

and

$$\begin{aligned}&\Delta(\phi_{c_m},\phi_{c_m})(S_t,0)\\ &=\int h_{\sigma_tS_t}(y)\phi_{c_m}^2(S_t,y)dy-\left(\int h_{\sigma_tS_t}(y)\phi_{c_m}(S_t,y)dy\right)^2\\ &=\int h_{\sigma_tS_t}(y)\left[\left(\frac{y}{S_t}\right)^4 I_{A_m}+\phi_{c_m}^2(S_t,y)I_{A_m^c}\right]dy\\ &\quad-\left(\int h_{\sigma_tS_t}(y)\left[\left(\frac{y}{S_t}\right)^2 I_{A_m}+\phi_{c_m}(S_t,y)I_{A_m^c}\right]dy\right)^2\\ &=\left[\int h_{\sigma_tS_t}(y)\left(\frac{y}{S_t}\right)^4dy-\left(\int h_{\sigma_tS_t}(y)\left(\frac{y}{S_t}\right)^2dy\right)^2\right]I_{A_m}\\ &\quad+\left[\int h_{\sigma_tS_t}(y)\phi_{c_m}(S_t,y)^2dy-\left(\int h_{\sigma_tS_t}(y)\phi_{c_m}(S_t,y)dy\right)^2\right]I_{A_m^c};\end{aligned}$$

hence

$$
\begin{aligned}
&\Delta(\phi_{c_m},\phi_{c_m})(S_t,0)^{1/2} \\
&=\left[\int h_{\sigma_t S_t}(y)\left(\frac{y}{S_t}\right)^4 dy-\left(\int h_{\sigma_t S_t}(y)\left(\frac{y}{S_t}\right)^2 dy\right)^2\right]^{1/2} I_{A_m} \\
&\quad+\left[\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)^2 dy-\left(\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)dy\right)^2\right]^{1/2} I_{A_m^c} \\
&=(2\sigma_t^4)^{1/2} I_{A_m}+\left[\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)^2 dy-\left(\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)dy\right)^2\right]^{1/2} I_{A_m^c}.
\end{aligned} \tag{A.11}
$$

Plug (A.10) and (A.11) into (A.9), and note that by the assumption that $\beta_n = O(n^{-\gamma})$,

$$
\sqrt{n}\frac{\beta_n^2}{\pi^2}\int_0^1 \frac{1}{S_t^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\exp\left(-2\pi^2 k^2\frac{\sigma_t^2 S_t^2}{\beta_n^2}\right)dt \to 0 \text{ a.s. on } A_m \text{ as } n\to\infty.
$$

One has,

$$
\begin{aligned}
&\sqrt{n}\left[U(n,\phi_{c_m})-\left(\int_0^1 \sigma_t^2 dt+\frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt\right)\right] I_{A_m} \\
&+\sqrt{n}\left[U(n,\phi_{c_m})-\int_0^1 \Gamma\phi_{c_m}(S_t,\beta_n)dt\right] I_{A_m^c} \\
&\to_{\textit{stably in law}} \\
&\quad ZI_{A_m}+\int_0^1\left[\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)^2 dy-\left(\int h_{\sigma_t S_t}(y)\phi_{c_m}(S_t,y)dy\right)^2\right]^{1/2} dB_s I_{A_m^c},
\end{aligned}
$$

where $Z \sim \int_0^1 (2\sigma_t^4)^{1/2} dB_s$, $B \perp\!\!\!\perp W$.

For any continuous function g that vanishes outside a compact set, the above stable convergence implies that $\forall E\in\mathcal{F}$,

$$
\begin{aligned}
&E\left[g\left(\sqrt{n}\left[U(n,\phi_{c_m})-\left(\int_0^1 \sigma_t^2 dt+\frac{\beta_n^2}{6}\int_0^1 \frac{1}{S_t^2}dt\right)\right] I_{A_m}\right) I_{A_m} I_E\right] \\
&\to E\left[g\left(\int_0^1 (2\sigma_t^4)^{1/2} dB_s I_{A_m}\right) I_{A_m} I_E\right].
\end{aligned} \tag{A.12}
$$

And by defining $\eta_{c_m}(\cdot,\cdot)$ to be

$$\eta_{c_m}(x,y)=\begin{cases}\left(\frac{1}{x}\right)^2, \text{ when } x\geq c_m;\\ \left(\frac{3}{{c_m}^4}x^2-\frac{8}{{c_m}^3}x+\frac{6}{{c_m}^2}\right), \text{ when } x<c_m,\end{cases}$$

one has,

$$V_0^n I_{A_m}=V^n I_{A_m}-\frac{\beta_n^2}{6}U(n,\eta_{c_m})I_{A_m}. \tag{A.13}$$

Again, by Theorem 3.1 of Delattre and Jacod (1997),

$$U(n,\eta_{c_m})I_{A_m}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{(S_{i/n}^{(\alpha_n)})^2}I_{A_m}\rightarrow_P\int_0^1\frac{1}{S_t^2}dtI_{A_m}.$$

and

$$\sqrt{n}\left(\frac{\beta_n^2}{6}U(n,\eta_{c_m})-\frac{\beta_n^2}{6}\int_0^1\frac{1}{S_t^2}dt\right)I_{A_m}=O_P(\beta_n^2)=o_P(1). \tag{A.14}$$

By (A.8), (A.13), and (A.14),

$$\sqrt{n}V_0^n I_{A_m}=\sqrt{n}(U(n,\phi_{c_m})-\frac{\beta_n^2}{6}\int_0^1\frac{1}{S_t^2}dt)I_{A_m}+o_p(1).$$

Also since that g is uniformly continuous, $\forall E\in\mathcal{F}$,

$$\begin{aligned}&\lim_{n\to\infty}E\left[g\left(\sqrt{n}(V_0^n-\int_0^1\sigma_t^2dt)I_{A_m}\right)I_{A_m}I_E\right]\\ &=\lim_{n\to\infty}E[g(\sqrt{n}\left[U(n,\phi_{c_m})-\left(\int_0^1\sigma_t^2dt+\frac{\beta_n^2}{6}\int_0^1\frac{1}{S_t^2}dt\right)\right]I_{A_m})I_{A_m}I_E]\\ &=E\left[g\left(\int_0^1(2\sigma_t^4)^{1/2}dB_tI_{A_m}\right)I_{A_m}I_E\right] \text{ (by (A.12))},\end{aligned}$$

which implies, for any $\epsilon>0$, there exists N, such that $\forall n\geq N$,

$$\left|E\left[g(\sqrt{n}[V_0^n-\sigma_t^2]I_{A_M})I_{A_M}I_E\right]-E\left[g\left(\int_0^1(2\sigma_t^4)^{1/2}dB_tI_{A_M}\right)I_{A_M}I_E\right]\right|<\epsilon.$$

Note also that g is bounded, suppose $|g|\leq M_g$. Recall that $P(A_M^c)\to 0$, one can choose M such that $P(A_M^c)<\epsilon/M_g$.

So for $n \geq N$,

$$\left| E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt])I_E] - E\left[g\left(\int_0^1 (2\sigma_t^4)^{1/2} dB_t\right) I_E\right]\right|$$

$$\leq \left| E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt]I_{A_M})I_{A_M} I_E] - E\left[g\left(\int_0^1 (2\sigma_t^4)^{1/2} dB_t I_{A_M}\right) I_{A_M} I_E\right]\right|$$

$$+ 2M_g * P(A_M^c) \leq 3\epsilon$$

Hence we've proved that for all continuous function g that vanishes outside a compact set, $\forall E \in \mathcal{F}$,

$$\lim_{n\to\infty} E[g(\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt])I_E] = E\left[g\left(\int_0^1 (2\sigma_t^4)^{1/2} dB_t\right) I_E\right],$$

i.e.,

$$\sqrt{n}[V_0^n - \int_0^1 \sigma_t^2 dt] \rightarrow_{\mathcal{L}-stably} \int_0^1 (2\sigma_t^4)^{1/2} dB_t.$$

This finishes the proof of Theorem 3. The proof of Theorem 2 is basically contained in the proof above.

A.4 The Case of General μ_t and σ_t

Step 1: For general cases when $\mu_t \neq 0$, if there exists $L_\sigma, U_\sigma, C_\mu \in (0, \infty)$, such that $L_\sigma \leq \sigma_t \leq U_\sigma$ and $|\mu_t| \leq C_\mu$ for $t \in [0, 1]$, the previous results all hold.

For the simplicity of notation, we consider the log scale. Let P be the probability measure corresponding to the system

$$dX_t = \sigma_t dW_t$$

and Q the probability measure corresponding to the system

$$dX_t = \mu_t dt + \sigma_t dW_t^Q,$$

where W_t and W_t^Q are standard Brownian motions under P and Q respectively.

Note that by the Girsanov Theorem (see, for example, page 164 of Øksendal (2003)), for bounded σ_t and μ_t (as stated in the conditions of "Step 1"), P and Q are mutually absolutely continuous.

The following proposition justifies the conclusion of "Step 1".

Proposition 1 (Mykland and Zhang (2009)) *Suppose that ζ_n is a sequence of random variables which converges stably to $N(b,a^2)$ under P (meaning that $N(b,a^2)=b+aN(0,1)$, where $N(0,1)$ is a standard normal variable independent of $\mathcal{F}$, also a and b are $\mathcal{F}$ measurable). Then ζ_n converges stably in law to $b+aN(0,1)$ under Q, where $N(0,1)$ remains independent of $\mathcal{F}$ under Q.*

Step 2: for locally bounded σ_t and μ_t, the stable convergence and the convergence in probability stay valid.

This can be proved by a localization argument which uses essentially the same techniques as in the derivation in the last part of Section A.3. For example, to unbound σ_t, one considers a sequence of stopping times τ_m corresponding to a sequence of positive constants σ_m which increases to infinity as $m \to \infty$: $\tau_m = min\{t : \sigma_t^2 \geq \sigma_m^2\}$, and note the fact that the sets $\{\tau_m > T\} \nearrow \Omega$.

In particular, the locally bounded assumption is automatically satisfied when σ_t and μ_t are continuous.

A.5 Proof of Theorem 4

Similar argument as the Proof of Theorem 3 in Li and Mykland (2007) gives the result.

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