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Jumps in equilibrium prices and market microstructure noise

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ABSTRACT

Asset prices observed in financial markets combine equilibrium prices and market microstructure noise. In this paper, we study how to tell apart large shifts in equilibrium prices from noise using high frequency data. We propose a new nonparametric test which allows us to asymptotically remove the noise from observable price data and to discover jumps in fundamental asset values. We provide its asymptotic distribution to decide when such jumps occur. In finite samples, our test offers reasonable power for distinguishing between noise and jumps. Empirical evidence indicates that it is necessary to incorporate the presence of jumps in equilibrium prices.

1. Introduction

Asset prices observed in financial markets are determined by two important, unobservable components. One is equilibrium prices, which reflect demand and supply of assets. These are also called as efficient prices, incorporating investors' thoughts on market information. The other is market microstructure noise induced by the frictions with which actual trades take place. Examples of such frictions are tick size, discrete observation, bid–ask spread, and other trading mechanics.[1] Given that both components are essential ingredients for trading, as indicated in Black (1986), researchers have sought a better understanding of both components and of their interactions. In particular, in recent years, with the availability of databases consisting of observations sampled at ultra-high frequency up to every second, extensive research that takes advantage of such data for better volatility and noise estimation has appeared, and the economic implications of volatility and noise have also been investigated in many studies.[2]

In this paper, we are motivated to question the assumptions imposed by most of the aforementioned studies for log equilibrium prices to follow diffusion processes. Although it is simpler to study this issue under such assumptions, it is widely known in the asset pricing literature that financial markets experience jumps in prices that are too large to be explained by pure diffusion processes, and their presence has been incorporated in numerous theoretical and empirical studies.[3] Obviously, one can argue that all the evidence of jumps documented in the previous asset pricing literature based on discretely sampled data is due to noise and hence, a diffusion assumption for efficient prices would be valid since noise indeed creates discreteness in recorded prices and it is thus difficult to tell through existing empirical methods if there are fundamental shifts in underlying asset values.[4] Nonetheless, distinguishing

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1 Other examples include institutional structure, transaction costs, adverse selection due to asymmetric information for different traders, trading size, volume, liquidity, and dealer's inventory control, among others. [See O'Hara (1995) and Hasbrouck (2004) and the references therein.]

2 See Aït-Sahalia et al. (2011), Bandi and Russell (2006), Zhang et al. (2005), and Hansen and Lunde (2006).

3 See Bates (1996), Bakshi et al. (1997), Aït-Sahalia (2002), Andersen et al. (2002), Pan (2002), Chernov et al. (2003), Eraker et al. (2003), and Johannes (2004).

4 Many empirical methods for testing jumps in asset prices with high frequency observations do not take into consideration the presence of market microstructure noise. See Barndorff-Nielsen and Shephard (2006), Aït-Sahalia and Jacod (2009), Mancini (2001), and Lee and Mykland (2008). Andersen et al. (2007) considered a case with i.i.d. noise.

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efficient price jumps from noise is important, first because, if there were in fact dramatic changes (jumps) in the fundamental values of underlying asset prices but these were neglected, as noted in various studies, their implications for financial management such as pricing and hedging would be significant. Second, we believe that distinguishing jumps in efficient prices from noise and understanding their interactions should give us a better tool for event studies, which we often employ in empirical investigations of market trading behavior.

Specifically, we propose a new empirical test that suggests preprocessing price level data for the purpose of de-noising as well as making a distinction between jumps in efficient prices and noise. Assuming that noise has an additive effect on equilibrium prices, we first take local averages of observed prices over an upcoming local window in the preprocessing. This local averaging allows us to asymptotically remove the noise and approximate the true underlying prices. (The device has been studied by Jacod et al. (2009) and Podolskij and Vetter (2009) for estimating volatility). Therefore, evidence based on this test becomes about efficient prices. In order for econometricians to determine the rejection regions for claiming jump arrivals, we offer a limiting distribution of our test statistics. To execute the test, we need to input the variance of noise process and the volatility of return process. We suggest a noise variance estimator that is asymptotically immune to the presence of jumps in efficient prices and dependence in the noise process, and we use an existing volatility estimator by Podolskij and Vetter (2009) for the return process.

Our theory directly extends to $(k-1)$-dependent noise. For dependent noise, we suggest empirically determining the dependence, subsampling every k observations, and then applying the proposed techniques. Our test is designed to take full advantage of an ultra-high frequency database. Hence, as long as high frequency price data are available for analysis, they can be used to determine the behavior of both unobservable price processes and noise processes for any type of financial assets. In addition, the outcome of our test is robust to model specification because the suggested procedure is nonparametric.

After presenting asymptotic theories of inference, we discuss finite sample performance using Monte Carlo simulations. We present the size and power properties of our test and show that detectable jumps tend to depend on the magnitude of noise variance. When the noise variance level is high (low), the test tends to detect jumps that are greater (smaller) in size. For a given jump size, however, we can increase the power of the test by increasing the frequency of observations over a fixed time interval.

Finally, we apply our new test of jumps in equilibrium prices and estimation procedure for noise variance to August 2007 IBM stock trade data from the TAQ database. In order for the asymptotic results of theoretical inference to be most effective in the data analysis, we use all tick-by-tick data available sampled at the highest frequencies. Noise variance estimates for IBM trades are around 0.01%–0.03% on average and found to be greater at opening time (09:30–10:00) on trading days. Based on our new jump test, which takes into account the general form of dependent noise in the market, we strongly reject the null hypothesis of no jump models for equilibrium prices, which suggests evidence in favor of equity pricing models with jumps.

The rest of the paper is organized as follows. We start in Section 2 by setting up a theoretical framework for equilibrium prices and specify a model of microstructure noise due to market imperfection. In Section 3, we explain the intuition behind the development of our test and introduce its definition. In Section 4, we discuss the asymptotic behavior of our test and the noise variance estimator. Section 5 illustrates the finite sample performance of the noise variance estimator and of our test under general assumptions on noise. After discussing our empirical study in Section 6, we conclude in Section 7. All the proofs are in the Appendix.

2. Theoretical model

This section sets up a theoretical framework to test the presence of jumps in equilibrium prices, using market price data which include noise from market microstructure. We first fix a complete probability space $(\Omega, \mathcal{F}_t, \mathcal{P})$, where Ω is the set of events in a financial market, $\{\mathcal{F}_t : t \in [0, T]\}$ is right-continuous information filtration for market participants, and $\mathcal{P}$ is a data-generating measure.

We denote as $P(t)$ the unobservable log-equilibrium price at t, in which we test the presence of jumps. Under the null hypothesis, the continuously compounded return $dP(t)$ is represented as

$$dP(t) = \sigma dW(t), \tag{1}$$

where $W(t)$ is an $\mathcal{F}_t$-adapted standard Brownian Motion, so that the underlying process is an Itô process that has continuous sample paths. Under the alternative hypothesis with the presence of jumps, the return is characterized by a jump diffusion process as

$$dP(t) = \sigma dW(t) + Y(t)dJ(t), \tag{2}$$

where $dJ(t)$ is a Poisson-type jump process with a stochastic intensity of $\lambda(t)$ independent of $W(t)$. The $dJ(t)$ term is an indicator of jump arrival. This $P(t)$ describes the asset price evolution under a perfectly frictionless market, where there is costless trading or an infinitely liquid market.

For simplicity, we set the drift μ to 0. This does not affect the generality of our theoretical asymptotic results, cf. the discussion in Section 2.2 (pp. 1407–1409) of Mykland and Zhang (2009), which in turn builds on Girsanov's Theorem (see, e.g., Karatzas and Shreve (1991)).[5]

Econometricians observe market data for the above process through either quoted or transaction prices under market friction due to physical limits on observing data only at discrete times or to various other types of market noise. The transaction or quote price observed at t_i, denoted as $\widetilde{P}(t_i)$ in this paper, is determined by the efficient price $P(t_i)$ as well as by market microstructure noise $U(t_i)$. As in most of the empirical and theoretical market microstructure literature including Black (1986) and Stoll (2000), among others, we take a model with additive effect of noise on log equilibrium prices, so that

$$\widetilde{P}(t_i) = P(t_i) + U(t_i). \tag{3}$$

Throughout this paper, we impose the following assumptions on observation times, latent price process, and noise.

Assumption A.

A.1: Ultra high frequency observation times.

We set the full grids $\mathcal{G}_n$ over the fixed time horizon $[0, T]$. Each observation time is set as $t_i = t_{n,i}$ and belongs to $\mathcal{G}_n = \{0 = t_{n,0} < t_{n,1} < \cdots < t_{n,n} = T\}$. The distance between two successive observations, $\Delta t_{n,i} = t_{n,i} - t_{n,i-1}$, is not necessarily fixed and can change over time depending on i. We assume

$$\max_{1\le i\le n} |\Delta t_{n,i}| = O_p(n^{-\beta}) \quad \text{for some } \beta > \frac{1}{2}, \tag{4}$$

so that the grid becomes dense in $[0, T]$ as $n \to \infty$. The subscript n is normally suppressed in our discussion.[6] We note that most

[5] In addition, from the empirical data analysis standpoint, we note, as in our earlier paper (Lee and Mykland, 2008), that estimating μ may introduce additional standard error. This would seem to be corroborated (in a different setting) by the discussion in Section 4.2 (Remark 8 and Fig. 1, pp. 1423–1424) of Mykland and Zhang (2009). We discuss in the Appendix a modified statistic in the presence of nonzero drift, with which our main result continues to hold.

[6] We use O_p notation throughout this paper to mean that for random vectors $\{X_n\}$ and non-negative random variable $\{d_n\}$, $X_n = O_p(d_n)$ if for each $\epsilon > 0$, there exists a finite constant M_ϵ such that $P(|X_n| > M_\epsilon d_n) < \epsilon$ eventually.

existing theory assumes that $\beta = 1$ (which, in particular, is what happens in the case of equally spaced data), so this is a substantial weakening of standard conditions. Our condition also covers observation times $t_{n,i}$ coming from a Poisson process.

A.2: Equilibrium price process.

Volatility σ is constant over $[0, T]$. Jump sizes Y at jump times within $[0, T]$ are independent and identically distributed and have mean μ_y and standard deviation σ_y.

A.3: Market microstructure noise. The noise distribution is stationary and given by

$$U(t_i) \sim_{\mathcal{D}} (0, q^2), \tag{5}$$

by which we mean that the noise follows a general process with its mean 0 and standard deviation q, which is also called market quality parameter or effective spread. We further assume that $E(U(t_i)^4) < \infty$.

Remark 1 (*Dependent Noise*)**.** Our theory directly extends to $(k-1)$-dependent noise $U(t_i)$. In applications, we suggest empirically determining the dependence $k-1$ and collecting every kth observation to create $\mathcal{G}_n^k = \{0 = t_{n,0} < t_{n,k} < t_{n,2k} < \cdots\}$. This reduces the problem to independent observations. For this reason, the theoretical results are written as if observations were independent, after subsampling. A slightly more elaborate theory would permit the sampling of every observation, and the market quality parameter would then take the form

$$(q')^2 = \mathrm{Var}(U(t_0)) + 2\sum_{i=1}^{k} \mathrm{Cov}(U(t_0), U(t_i)). \tag{6}$$

Assumption A.1 implies that the distance between two successive observations can be irregular, which is the usual characteristic of ultra high frequency data, for example, data available in the TAQ database. Although we take σ as constant in assumption A.2, most likely a similar result holds for time-varying σ, and this is certainly the case when the $U(t_i)$'s are normally distributed. The motivation for imposing Assumption A.3 is to allow a dependent structure for general noise so that we cover most of the models found in the market microstructure literature. q in Assumption A.3 describes how noisy the market is. $q = 0$ is equivalent to a frictionless market where equilibrium prices $P(t)$ can be observed. Thus, q represents the degree of market imperfection or the quality of trading exchange. Approximately, if we use a mid-point quote as the observed price, we can interpret the magnitude of noise as the difference between the mid-point quote and the corresponding equilibrium price. The justification can also be found in Hasbrouck (2004) and the references therein.

3. Intuition and definition of the test

This section explains the intuition behind the development of our test and its definition. In order to understand the interaction between jumps in equilibrium prices and microstructure noise, we first consider the null hypothesis, whereby there is no jump in the equilibrium price process, as in Eq. (1), and we observe its data with noise. Suppose econometricians calculate the log returns using recorded prices at high frequency. As the distance between two successive observation time stamps gets smaller (and our observation time becomes closer to continuous time: $\max_{0\le i\le n} |\Delta t_{n,i}| \to 0$), the statistics based on these observed log returns will be about noise, not about the latent price process. This is because noise, for example bid–ask spread, does not disappear in observed prices even if $\max_{0\le i\le n} |\Delta t_{n,i}| \to 0$. However, the effect of the Brownian motion process disappears theoretically. In other words, noise plays a dominant role at such high frequencies.[7]

Now, how about the alternative hypothesis whereby there are jumps in equilibrium prices as in Eq. (2) and we observe data under the alternative hypothesis with noise? As before, the effect of the Brownian motion disappears as $\max_{0\le i\le n} |\Delta t_{n,i}| \to 0$. But this time, two kinds of discreteness remain in the observed returns. One is noise, as explained above, and the other is jumps in latent equilibrium prices. Even if $\max_{0\le i\le n} |\Delta_{n,i}| \to 0$, neither will disappear both theoretically and empirically. This is where the distinction becomes problematic because when we have very large changes in observed prices, this could be due to noise or to jumps in efficient prices.

In order to tell jumps in equilibrium prices from noise, we suggest preprocessing the raw price level data. Instead of using observed prices directly for return calculation, we first average observed prices over an upcoming block of size M. This technique of averaging observed prices with an appropriate M allows us to asymptotically remove the noise from the price data which are contaminated by the noise and to extract the level of equilibrium prices. These price levels preprocessed from nonoverlapping blocks are used in our test statistics to determine the presence of jumps.[8] Formally, we write the preprocessing procedure and the test statistic for jumps in equilibrium prices as in Definition 1.

Definition 1. Let $M = M_n$ be the block size such that

$$M \sim C\lfloor n/k \rfloor^{\frac{1}{2}} \tag{7}$$

as $n \to \infty$. The preprocessed price for de-noising, $\widehat{P}(t_j)$, is the averaged log price over the block of size M such that $\widehat{P}(t_j) = \frac{1}{M}\sum_{i=\lfloor j/k\rfloor}^{\lfloor j/k\rfloor+M-1} \widetilde{P}(t_{ik})$, where $\widetilde{P}(t_{ik})$ is the log price data from $\mathcal{G}_n^k$, subsampled due to the $(k-1)$-dependent noise. Then, we again sample $\widehat{P}(t_j)$'s at every M observations from $\mathcal{G}_n^k$. The grid for this subsample is set as $\mathcal{G}_n^{kM} = \{t_{n,0} < t_{n,kM} < t_{n,2kM} < \cdots\} = \{t_0 < t_{kM} < t_{2kM} < \cdots\} \subset \mathcal{G}_n$. To test the presence of jumps in equilibrium price between t_{j+kM} to t_j, the statistic $\mathcal{L}(t_j)$ is defined as

$$\mathcal{L}(t_j) \equiv \widehat{P}(t_{j+kM}) - \widehat{P}(t_j) \tag{8}$$

with the observation time $t_j \in \mathcal{G}_n^{kM}$ for all j.

4. Theory of inference for equilibrium price with noise

In this section, we study inference theory. Results are discussed with a fixed market quality parameter q and volatility σ of asset returns. We carry out our formal study with this simplified assumption on noise and volatility as a first step to theoretically refine our understanding. In Section 5, we ensure that the results hold in more realistic conditions such as time-varying noise or stochastic volatility through simulation studies.

4.1. Asymptotic behavior of the test

In this subsection, we discuss the asymptotic behavior of our test statistic and how to set up the rejection region to detect jumps in equilibrium prices.

7 This is noted in Zhang et al. (2005) and Bandi and Russell (2006), suggesting not using most frequently observed returns but using less frequently observed returns in order to make a better volatility $\sigma(t)$ estimation. These studies also offer optimal sampling frequency for sample selection. However, they assume that there is no jump in equilibrium prices.

8 This pre-averaging technique has been proposed for volatility estimation for diffusion processes in the presence of noise. [See Jacod et al. (2009) and the references therein.]

In order to simplify our discussion, we standardize the test statistic $\mathcal{L}(t_j)$. Notice that under our assumptions, the expected value of $\mathcal{L}(t_j)$, $E[\mathcal{L}(t_j)] = 0$, and its scaled variance $V_n = \mathrm{Var}[\sqrt{M}\mathcal{L}(t_j)]$ has its limit

$$\operatorname*{plim}_{n\to\infty} V_n = \frac{2}{3}\sigma^2 C^2 T + 2q^2, \quad (9)$$

where C is as in Definition 1. Here, we obtain the following lemma for the standardized test statistic $\mathcal{X}(t_j)$.

Lemma 1. *Under Assumptions* A.1–A.3 *and Eq.* (7)–(8), *also suppose that there is no jump in efficient prices under the null hypothesis, as in Eq.* (1). *For any given* $t_j \in \mathcal{G}_n^{kM}$, *we set*

$$\mathcal{X}(t_j) = \frac{\sqrt{M}}{\sqrt{V_n}}\mathcal{L}(t_j), \quad (10)$$

where $V_n = \mathrm{Var}[\sqrt{M}\mathcal{L}(t_j)]$. *Then, as* $n \to \infty$,

$$\mathcal{X}(t_j) \xrightarrow{\mathcal{D}} \mathcal{N}(0,1), \quad (11)$$

where $\mathcal{N}(0,1)$ *denotes a standard normal random variable.*

The above lemma states that the differences in averaged log prices become Gaussian in the limit.[9] Given this important result, we study in Theorem 1 below the distribution of the maximums of $|\mathcal{X}(t_j)|$ to determine the rejection region of our test.

Theorem 1. *Under assumptions* A.1–A.3 *and Eqs.* (7)–(10), *also suppose that there is no jump in efficient prices under the null hypothesis, as in Eq.* (1). *Then, as* $n \to \infty$,

$$\frac{\max_{t_j\in\mathcal{G}_n^{kM}} |\mathcal{X}(t_j)| - A_n}{B_n} \xrightarrow{\mathcal{D}} \xi, \quad (12)$$

where ξ *follows a standard Gumbel distribution whose cumulative distribution function* $P(\xi \le x) = \exp(-e^{-x})$,[10]

$$A_n = \left(2\log\left\lfloor\frac{n}{kM}\right\rfloor\right)^{1/2} - \frac{\log\pi + \log\left(\log\left\lfloor\frac{n}{kM}\right\rfloor\right)}{2\left(2\log\left\lfloor\frac{n}{kM}\right\rfloor\right)^{1/2}}, \quad \text{and}$$
$$B_n = \frac{1}{\left(2\log\left\lfloor\frac{n}{kM}\right\rfloor\right)^{1/2}}. \quad (13)$$

Therefore, as $n \to \infty$,

$$\hat{\xi}_n = B_n^{-1}\left(\frac{\sqrt{M}}{\sqrt{V_n}}\max_{t_j\in\mathcal{G}_n^{kM}}|\mathcal{L}(t_j)| - A_n\right) \xrightarrow{\mathcal{D}} \xi, \quad (14)$$

where ξ *is as in Eq.* (12) *and* A_n *and* B_n *are as in Eq.* (13).

Specifically, the above main theorem implies that in the presence of noise, one can find maximum of the absolute differences in averaged log prices (i.e., maximum among $|\mathcal{L}(t_j)|$'s with $t_j \in \mathcal{G}_n^{kM}$) and use the Gumbel distribution to select the rejection region for the maximum.[11] For example, if we set the significance level at 1%, then the threshold for rejecting the null hypothesis using ξ_n can be found from the 99th percentile of the standard Gumbel distribution. Now, we study in Theorem 2 below how this test would react to jumps in equilibrium prices.

Theorem 2. *Under Assumptions* A.1–A.3 *and Eqs.* (7)–(10), *also suppose that there can be jumps in efficient prices under the alternative hypothesis, as in Eq.* (2). *If there are F jumps at times* $\tau_f \in [0,T]$ *for a finite F, then,*

$$\max_{t_j\in\mathcal{G}_n^{kM}} |\mathcal{L}(t_j)| = \max_{1\le f\le F} |Y(\tau_f)| + o_p(1), \quad (15)$$

where $Y(\tau_f)$ *is the equilibrium price jump size at jump time* τ_f.

As stated in Theorem 2, under the alternative hypothesis, the test statistic would be close to the maximum jump size over the interval within which we would like to test the jumps in equilibrium prices. Notice that, by Lemma 1, each quantity defined in Eq. (8) converges to zero under the null hypothesis of no jump. Furthermore, $\max_{t_j\in\mathcal{G}_n^{kM}} |\mathcal{L}(t_j)| = O_p(A_n/\sqrt{M})$ also converges to zero under the null hypothesis. Therefore, this test will detect the presence of jumps (which can be single or multiple) in the interval under consideration.

4.2. Consistent estimation of noise variance in the presence of jumps

As can be noticed in Theorem 1, in order to apply our test, we need a consistent estimator for V_n, $\widehat{V}_n$. Based on Eq. (9), we suggest using $\widehat{V}_n = \frac{2}{3}\hat{\sigma}^2 C^2 T + 2\hat{q}^2$, where $\hat{\sigma}$ and $\hat{q}$ are consistent estimators for volatility σ and noise variance q, respectively. For $\hat{\sigma}$, we suggest using the estimator proposed by Podolskij and Vetter (2009), who proposed a volatility estimator that is robust to the presence of jumps and noise.[12] For estimating noise variance q, we suggest in the following proposition a new estimator that takes into account the dependent noise, which can be used regardless of the presence of jumps in efficient prices.

Proposition 1. *Suppose that the noise follows a* $(k-1)$*-dependent process with* $1 \le k < \infty$. *Its variance estimator over the interval* $[0,T]$, $\widehat{\mathcal{Q}(k)}$, *is defined as*

$$\widehat{\mathcal{Q}(k)} \equiv \left(\frac{1}{n'}\sum_{m=1}^{n'}(\widetilde{P}(t_m) - \widetilde{P}(t_{m+k}))^2\right)^{1/2}, \quad \text{with } n' = n-k. \quad (16)$$

Then, regardless of the presence of jumps, as $\max_i |\Delta t_i|$ *goes to 0,*

$$\widehat{\mathcal{Q}(k)} \xrightarrow{P} \sqrt{2}q. \quad (17)$$

Therefore, q can be consistently estimated by $\hat{q} = \widehat{\mathcal{Q}(k)}/\sqrt{2}$.

Under both hypotheses on the presence of jumps, this realized power variation estimator does not converge in probability to the integrated variance of returns from efficient prices. Rather, it converges to a quantity that explains variance in noise.

As can be seen, our analysis depends on the order of noise dependence k, which is not observable and unknown in practice. The k can be estimated in the following way. Assuming that

9 Notice that the numerator of this test statistic is the difference in averaged log prices, which is a crucial component in distinguishing jumps in equilibrium prices from noise. As we discussed earlier, the difficulty in this distinction comes from the fact that jumps and noise have the same asymptotic order. In that sense, the design of our test is better than that of Bos et al. (2009), who used the observed log return (without de-noising) in their jump test statistic.

10 This standard Gumbel distribution has its probability density function $P(\xi = x) = e^{-x}\exp(-e^{-x})$ with the mean Euler–Mascheroni constant approximately 0.577 and standard deviation $\pi/\sqrt{6} \approx 1.2825$.

11 A similar lemma was used in Lee and Mykland (2008), which does not take the presence of noise into account for their jump detection.

12 See Subsection 3.1.2 of Podolskij and Vetter (2009) for details. We use this robust estimator because the observed data could be contaminated by noise and the efficient prices could experience jumps. Regardless of their presence, volatility σ should be consistently estimated.

observed log returns sampled at the highest frequency give us information about noise, we suggest calculating their serial correlation function and determine the number of dependence lags by applying the usual significance test for this autocorrelation. Our simulation study presented in Section 5 indicates that this method of k selection works well in the presence of jumps. In practice, since the noise distribution is not known, we suggest in this paper using this noise variance estimator, which does not impose any assumption on its distribution and with which the test works reasonably well in finite samples. See Section 5 for a more detailed illustration of its finite sample performance.

5. Simulation for finite sample behavior

Our asymptotic arguments require infinite sampling, which is not completely achieved in practice, though enough high frequency data are available due to recent advances in information technology. In this section, we examine by Monte Carlo simulation the finite sample performance of our test in terms of both size and power of the test. As shown, overall simulation results support our inference theory presented in Section 4. We also suggest optimal block sizes M for pre-averaging, depending on noise variance parameters. As also noted earlier, although our theory is developed with constant volatility σ, we consider general scenarios with time-varying noise and stochastic volatility in finite samples to ensure that our results hold under more realistic market conditions.

For generating equilibrium prices, we consider jump diffusion models, as in Eq. (2), with both constant and stochastic volatility. For constant volatility, we set $\sigma = 30\%$ per year, which is usual for the US equity markets. For stochastic volatility, we assume the (Heston, 1993) model, specified as

$$d\sigma^2(t) = \kappa(\bar{\varsigma} - \sigma^2(t))dt + \omega\sigma(t)dB(t), \qquad (18)$$

where $B(t)$ denotes a Brownian Motion. For κ, $\bar{\varsigma}$, and ω, we used the parameter estimates from equity markets reported by Li et al. (2008), Table 4: $\kappa = 0.0162$, $\bar{\varsigma} = 0.8465$, and $\omega = 0.1170$.[13]

Here, we discuss specifications for noise processes. For independent noise, we simply generate $U(t_i)$ from a normal distribution, $\mathcal{N}(0, q^2)$. However, as discussed in Engle and Sun (2006), a more realistic noise model should incorporate its various characteristics such as stationarity and crosscorrelation between noise and equilibrium prices. Because the information flow affects both components of transactions, for example, it is likely that market microstructure noise is correlated with equilibrium price changes. Price determination by adverse selection under asymmetric information can also create various type of noise dependence [see O'Hara (1995)].

In order to incorporate such general properties of dependent noise, we use the general noise model employed by Engle and Sun (2006). We use their parameter estimates for an individual US stock reported as significant at 5%. Specifically, the cross-correlated model we employ for our simulation, relating current and lagged innovation in equilibrium prices to noise, is

$$U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma(t)dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma(t)dW(s) + X(t_i), \qquad (19)$$

Table 1
Performance of the noise variance (q) estimator.[a]

σ_y	0	$1 \times \sigma(t)$	$2 \times \sigma(t)$	$3 \times \sigma(t)$
Constant volatility and independent noise with $k = 1$				
MSE	6.2026e−009	2.1191e−004	0.0034	0.0169
Ave $\hat{k}$	2.1000	1.0500	1.0600	1.0300
Stochastic volatility and independent noise with $k = 1$				
MSE	6.1746e−009	2.1254e−004	0.0035	0.0167
Ave $\hat{k}$	2.0100	1.0400	1.0500	1.0400
Constant volatility and dependent noise with $k = 3$				
MSE	9.8161e−009	7.0682e−004	0.0110	0.0566
Ave $\hat{k}$	2.4500	3.0100	3.000	2.9800
Stochastic volatility and dependent noise with $k = 3$				
MSE	1.0432e−008	6.9473e−004	0.0115	0.0544
Ave $\hat{k}$	2.5400	3.0100	3.000	2.9900

[a] This table presents the performance of the noise variance q estimators in terms of Mean Squared Errors (MSEs). Assuming that noise can follow a general dependent process, the estimator based on quadratic variation (QV) as defined in Proposition 1 is used for this table. The simulation design for generating efficient prices and noise processes can be found in Section 5. We report the results depending on $\hat{k}$, which also has to be selected in each simulation run. The procedure for selecting the order of dependence k is described in Section 5. $k = 1$ represents the independent noise and $k > 1$ represents the dependent noise. The averages of estimated parameter $\hat{k}$ for k are presented. The number of simulations is 6000. Four different levels of jump sizes relative to volatility level are considered. $\sigma_y = 0$ represents the case without jumps.

where $X(t_i)$ is a normal random variable with standard deviation q, and θ_0 and θ_1 are set at 0.0861 and 0.06, respectively.[14] Though Engle and Sun (2006) also have estimates for q, we consider q at three different levels in order to see the impact of noise magnitude on the performance of our test. These q's are chosen around the estimates reported by Aït-Sahalia et al. (2005) and Bandi and Russell (2006). In particular, we set the market quality parameter q at different levels such as $q = 0.01\%$, 0.05%, and 0.1%. To study finite sample properties in the following subsections, we add these two types of noise under both the null and the alternative hypotheses for efficient prices $P(t_i)$, as in Eq. (3).

5.1. Performance of the noise variance estimator

We now study the performance of our newly proposed noise variance estimator. As a nonparametric estimator for noise variance, quadratic variation has been suggested in Zhang et al. (2005) and Bandi and Russell (2006), among others, assuming that there is no jump in efficient price processes. In theory, this estimator can also be used in the presence of finitely many jumps in efficient prices and general dependent noise. In this subsection, we study by simulation how the quadratic variation (QV) as a noise variance estimator performs in finite samples.

We simulate 6000 series of efficient prices from a jump diffusion process over a day with five-second frequency for Table 1. The jump intensity is set at 5% per year, and we consider cases with no jump as well as jumps with three jump size standard deviations σ_y at one, two, and three times σ. $U(t_i)$ is assumed, as discussed earlier. The order of noise dependence is unknown in practice. We calculate the serial correlation of noise and select the number of dependence lags. We apply the usual significance test at the significance level of 5% for this autocorrelation.

Table 1 explicitly shows numerical values for the Mean Squared Errors (MSEs) of the noise variance (q) estimator. As can be seen

[13] For all series generation, we used the Euler–Maruyama Stochastic Differential Equation (SDE) discretization scheme in Kloeden and Platen (1992), an explicit order 0.5 strong and order 1.0 weak scheme. We discard the burn-in period – the first part of the whole series – to avoid the starting value effect every time we generate each series.

[14] We also consider a non-normal dependent noise model using a uniform distribution with standard deviation q, and obtain similar results. We omit reporting the results in order to save space.

Table 2
Size and power of the test under constant volatility.[a]

Frequency (nobs)	$\sigma_y = 0$	$\sigma_y = 0.15\%$	$\sigma_y = 0.30\%$	$\sigma_y = 0.45\%$
Market quality parameter ($q = 0.01\%$)				
3 s (1200)	0.043	0.990	0.999	1.000
2 s (1800)	0.039	1.000	1.000	1.000
1 s (3600)	0.039	1.000	1.000	1.000
Market quality parameter ($q = 0.05\%$)				
3 s (1200)	0.032	0.770	0.999	1.000
2 s (1800)	0.043	0.900	1.000	1.000
1 s (3600)	0.044	0.989	1.000	1.000
Market quality parameter ($q = 0.1\%$)				
3 s (1200)	0.036	0.210	0.869	0.997
2 s (1800)	0.043	0.289	0.960	0.999
1 s (3600)	0.066	0.891	1.000	1.000

[a] This table reports performance (size (under $\sigma_y = 0$) and power (under $\sigma_y \neq 0$)) of our test for jumps in equilibrium prices in the presence of noise. The equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$ with a constant volatility level $\sigma(t) = \sigma = 30\%$. The test is based on the observed data contaminated by noise $U(t_i)$ generated from the dependent noise model studied by Engle and Sun (2006). We use their parameter estimates reported as significant at 5%. In particular, we simulate noise series from $U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma dW(s) + X(t_i)$, where $X(t_i)$ is a normal variable with mean 0 and variance q^2, and θ_0 and θ_1 are set at their estimates, which are 0.0861 and 0.06, respectively. The market quality parameter q's are chosen at various levels around values shown in our empirical studies as well as Aït-Sahalia et al. (2005) and Bandi and Russell (2006). The number of simulations is 6000. σ_y in the table denotes the standard deviation of the jump size distribution. Significance level α used for detection is 5%. Results under the independent noise model with finite variance and non-normal dependent noise model are similar to the results reported in this table.

Table 3
Size and power of the test under stochastic volatility.[a]

Frequency (nobs)	$\sigma_y = 0$	$\sigma_y = 0.15\%$	$\sigma_y = 0.30\%$	$\sigma_y = 0.45\%$
Market quality parameter ($q = 0.01\%$)				
3 s (1200)	0.045	0.995	1.000	1.000
2 s (1800)	0.049	1.000	1.000	1.000
1 s (3600)	0.041	1.000	1.000	1.000
Market quality parameter ($q = 0.05\%$)				
3 s (1200)	0.028	0.787	0.999	1.000
2 s (1800)	0.038	0.900	1.000	1.000
1 s (3600)	0.077	0.989	1.000	1.000
Market quality parameter ($q = 0.1\%$)				
3 s (1200)	0.049	0.229	0.881	0.994
2 s (1800)	0.035	0.293	0.957	0.999
1 s (3600)	0.053	0.898	1.000	1.000

[a] This table reports performance (size (under $\sigma_y = 0$) and power (under $\sigma_y \neq 0$)) of our test for jumps in equilibrium prices in the presence of noise. The equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$ with a stochastic volatility model, as described in Section 5. The test is based on the observed data contaminated by noise $U(t_i)$ generated from the dependent noise model studied by Engle and Sun (2006). We use their parameter estimates reported as significant at 5%. In particular, we simulate noise series from $U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma dW(s) + X(t_i)$, where $X(t_i)$ is a normal variable with mean 0 and variance q^2, and θ_0 and θ_1 are set at their estimates, which are 0.0861 and 0.06, respectively. The market quality parameter q's are chosen at various levels around values shown in our empirical studies as well as Aït-Sahalia et al. (2005) and Bandi and Russell (2006). The number of simulations is 6000. σ_y in the table denotes the standard deviation of the jump size distribution. Significance level α used for detection is 5%. Results under the independent noise model with finite variance and non-normal dependent noise model are similar to the results reported in this table.

in the table, the method for selecting order of noise dependence works well in the presence of jumps, although it tends to over/underestimate the order in the absence of jumps. This bias in the order does not seriously influence the performance of the noise variance estimator, as can be seen in the table. As expected, we find that as the jump sizes becomes greater, the MSEs of this estimator increases.

5.2. Size and power of the test

To investigate the size of the new test, we generate the equilibrium prices from a diffusion process $dP(t) = \sigma(t)dW(t)$. n is the number of observations over one trading hour. In this study, the numbers of observations are chosen at $n = 1200, 1800$, and 3600, which is equivalent to sample observations at every 3-s, 2-s, and 1-s intervals. The number of simulations is 6000. In order to examine the power of the test, the equilibrium prices are generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$ with a standard deviation σ_y of jump size distribution. Significance level α used for detection is 5%.

We report the probability of rejecting the null hypothesis of no jump in price processes. Tables 2 and 3 include both size (under $\sigma_y = 0$) and power (under $\sigma_y \neq 0$) when the volatility is constant at 30% and stochastic, respectively. In applications using ultra high frequency data, it is important to check first whether any test detects the presence of jumps spuriously and does not detect microstructure noise as jumps, because as explained in our introduction and in the intuition behind our test, asymptotically, both noise and jumps can be regarded similarly. The results show appropriate size properties (presented in the columns under $\sigma_y = 0$) and does not present spurious detection problems.

The overall results regarding the power of the test (presented in the columns under $\sigma_y \neq 0$) indicate that detectable jumps in equilibrium prices depend on noise level. If the magnitude of noise is greater, detectable jump sizes in equilibrium prices are greater, and hence, the power for small sized jumps decreases. Our test is designed to be robust to dependence of noise and we prove its robustness in the finite sample performance. In particular, the dependence does not change the power of our test significantly.[15] We also find that increasing frequency helps to improve the performance of the test.

5.3. Comparison with Lee and Mykland (2008) in the presence of noise

In this subsection, we discuss the impact of noise on the jump test by Lee and Mykland (2008) that is not devised to be robust to noise and compare its performance with that of the new test proposed in this paper. In order to avoid any distortion in the analysis due to the presence of noise, it was suggested in that study using data sampled sparsely, for example, sampling data every 15 min or 30 min. Here, we consider what will happen to the test if we use it using data collected extremely frequently up to 1 s.

If data are sampled too frequently, their jump robust volatility estimator (based on the scaled bipower variation) that Lee and Mykland (2008) used in the denominator of their test, will no longer estimate the instantaneous volatility but rather estimate the noise variance (after scaling). Therefore, when there are small jumps in efficient prices relative to large noise at a particular time, the test will not be able to detect those small jumps. In other words, the detection power for the smaller jumps will be reduced if the noise level is relatively large. However, if the noise level is small, the presence of noise will not play a critical role. This evidence is illustrated in Table 4, which includes results from a simulation study comparing the performance of the two tests in the presence of noise. For this study, we consider a general simulation setup with stochastic volatility and dependent noise models, as before. The specific details for the simulation design can also be found in the table note.

15 We find similar results under independent noise but omit reporting the results in order to save space.

Table 4
Comparison with Lee and Mykland (2008).[a]

Frequency (nobs)	No jump	$\sigma_y = 0$	$\sigma_y = 1q$		$\sigma_y = 2q$		$\sigma_y = 3q$	
Tests	LM1	LM2	LM1	LM2	LM1	LM2	LM1	LM2
	Jump size relative to market quality parameter ($q = 0.05\%$)							
3 s (1200)	0.012	0.034	0.025	0.059	0.287	0.320	0.816	0.786
2 s (1800)	0.008	0.030	0.022	0.071	0.412	0.483	0.940	0.920
1 s (3600)	0.010	0.046	0.034	0.091	0.658	0.709	0.995	0.988
	Jump size relative to market quality parameter ($q = 0.5\%$)							
3 s (1200)	0.010	0.046	0.016	0.275	0.350	0.889	0.885	0.997
2 s (1800)	0.009	0.046	0.021	0.593	0.461	0.998	0.961	1.000
1 s (3600)	0.010	0.041	0.024	0.918	0.673	1.000	0.998	1.000

[a] This table reports the comparative performance of Lee and Mykland (2008), which is not devised to be robust to noise, and of our jump test in the presence of noise. In particular, we compare the probability of rejecting the null hypothesis of no jump. To save space, we call the test by Lee and Mykland (2008) "LM1" and our test proposed in this study "LM2". Both test results are based on the observed prices, which combine both efficient prices generated from a jump diffusion process $dP(t) = \sigma(t)dW(t) + Y(t)dJ(t)$ and noise $U(t_i)$, which is generated from the dependent noise model studied by Engle and Sun (2006). These authors estimated the model using tick-by-tick data on randomly picked US individual equities, and we use their parameter estimates reported as significant at 5%. In particular, we simulate noise series from $U(t_i) = \theta_0 \int_{t_{i-1}}^{t_i} \sigma dW(s) + \theta_1 \int_{t_{i-2}}^{t_i} \sigma dW(s) + X(t_i)$, where $X(t_i)$ is a normal variable with mean 0 and variance q^2, and θ_0 and θ_1 are set at 0.0861 and 0.06, respectively. The number of simulations is 6000. σ_y in the table denotes the standard deviation of the distribution of jump size $Y(t)$. Significance level α used for detection is 5%.

This comparison study suggests that when the noise variance level is large but jump sizes are small, our new test, which takes into account the presence of noise, outperforms the jump test by Lee and Mykland (2008) in general. However, if the noise variance level is small, we find that the marginal benefit of devising the test to be robust to noise tends to decrease.

5.4. Block size M selection for local pre-averaging

One important tuning parameter for practical applications of this test is a block size M, the number of prices to be averaged to denoise the observed prices. As is common in nonparametric inference methods, our test is also sensitive to this choice. In theory, it needs to satisfy the condition stated in Eq. (7), that is, $M \sim C\lfloor n/k \rfloor^{1/2}$. One way to choose this block size is to find an optimal constant C that satisfies this. This optimal constant can be easily searched by simulation, which can also ensure the proper finite sample performance of our test. For users' convenience, we list in Table 5 various possible q values and optimal C depending on q. This M does depend on q but not much on the dependence of the noise process. Therefore, the C term listed in the table can be used regardless of the dependence in noise. In general, when q is greater, we need larger block sizes for pre-averaging. This is natural because of the purpose of pre-averaging, which is to denoise the observed prices. Since we do not know q in practice, we should first estimate q using the estimator in Proposition 1 and determine C according to Table 5. We use this rule for our simulation and empirical analysis throughout this paper.

6. Empirical analysis for IBM stock trades

We apply our new procedures to observed price data from actual stock trades. In order to make our asymptotic results most

Table 5
Optimal block size M for preaveraging.[a]

q (%)	Optimal C	q (%)	Optimal C	q (%)	Optimal C
0.01	1/19	0.1	1/18	0.6	1/9
0.03	1/19	0.2	1/16	0.7	1/9
0.05	1/19	0.3	1/16	0.8	1/9
0.07	1/18	0.4	1/9	0.9	1/8
0.09	1/18	0.5	1/9	1.0	1/8

[a] This table presents the optimal value for parameter C for block size selection in $M \sim C\lfloor n/k \rfloor^{1/2}$ as a function of q. These optimal values are searched by minimizing the absolute distance between the true size and the empirical size of the test using simulated price data. The number of simulations is 6000 and the most general dependent noise model studied by Engle and Sun (2006) is used to generate noise.

effective in any data analysis, it is best to use all tick-by-tick transaction data sampled at the highest frequency.

6.1. Data

Data are collected from the TAQ database, and we only consider transactions on the New York Stock Exchange (NYSE) to be consistent in terms of trading mechanisms for all trades under investigation. The sample period is August 2007. Due to interrupted trading in the NYSE overnight, all trades before 9:30 am or after 4:00 pm are discarded. We also exclude the first trade after 9:30 am for each trading day, which is the usual way of avoiding the overnight effect [see Engle and Sun (2006), for example]. For trades that take place at the same time and hence have multiple prices at any given time, we take the averaged observed price, which removes all transactions with zero duration. We discard all recording errors such as zero prices (if any). In order to eliminate *bounce-back* type data errors as noted in Aït-Sahalia et al. (2011), we remove obvious outliers and only keep data with log returns within the range of 7 standard deviations around its mean. The total number of tick-by-tick observations used in our analysis is 167,595.

In Table 6, we include summary statistics for the number of trades, durations Δt_i in seconds, log returns $\Delta \log \tilde{P}(t_i)$ in basis points, and observed prices $\tilde{P}(t_i)$ in dollar terms. We have 23 trading days for August 2007 and 6.5 trading hours for each trading day. We take the time horizon for each test, T, to be one hour after 10:00 am till 4:00 pm and 30 min for opening half hours from 9:30 am till 10:00 am each day. Columns in Table 6, for example 11–12, include information about trades after 11 am (inclusive) and before 12 pm (exclusive). Though there is seasonality in the number of trades, we have a high enough number of trades within all horizons for our asymptotic results to be effective. Durations between two consecutive trades Δt_i have averages below 5 s. $\Delta \log \widetilde{P}(t_i)$ is the first difference in observed log prices sampled at the highest frequencies available.

6.2. Empirical results

We first discuss the estimated market quality parameter q, which is the standard deviation of the market microstructure noise process in Eq. (5). In order to determine $(k-1)$ for the serial dependence of noise, we calculate the autocorrelation of the observed log returns at the highest frequency for every horizon and apply the usual significance test at 5%, as in Fig. 1, to determine the number of dependent lags. Fig. 1 shows one representative sample autocorrelation function of the most frequently sampled log returns on August 1, 2007. The two solid horizontal lines in the graph for lags of 2 and beyond make the 95% confidence band. If the dot is inside the band, it means that the corresponding lag is not significant. We apply this rule for determining the order of dependence at each time of testing.

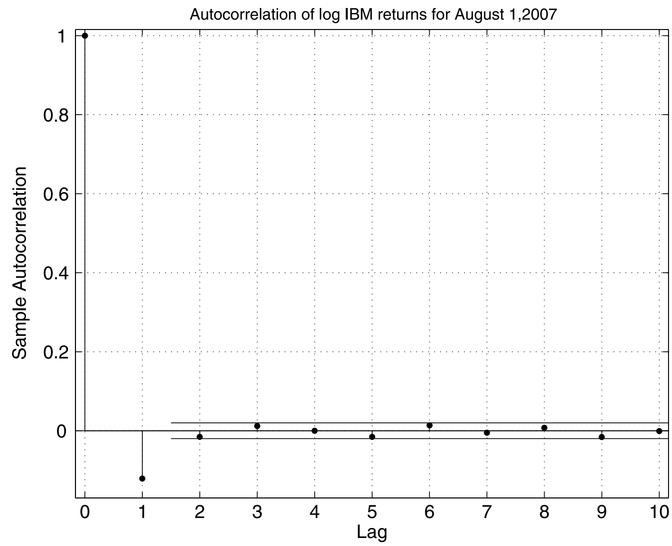

Fig. 1. Sample autocorrelation of IBM stock returns during August 2007. The figure includes a representative sample autocorrelation function of returns from IBM stocks traded on the New York Stock Exchange (NYSE). This graph is for August 1, 2007, and we have qualitatively similar figures for other days and hours during the whole month of August 2007. We calculate this sample autocorrelation of returns sampled at the highest frequency and employ the significant lag number for k in our analysis. The two solid horizontal lines in this graph for the lags of 2 and beyond make the 95% confidence band. If the dot is inside the band, this means that the corresponding lag is not significant.

Table 6
Descriptive statistics of IBM stock trades during August 2007.[a]

Trading hour	9:30–10	10–11	11–12	12–13	13–14	14–15	15–16
Min no. of trades	494	884	622	553	556	626	1052
Max no. of trades	1107	1783	1510	1750	1505	1814	2323
Ave no. of trades	677	1184	1024	875	848	1056	1622
Std no. of trades	147	274	257	268	228	283	352
Min Δt_i (s)	1	1	1	1	1	1	1
Max Δt_i (s)	35	60	59	61	63	69	41
Ave Δt_i (s)	2.701	3.180	3.729	4.397	4.512	3.646	2.324
Std Δt_i (s)	2.865	3.636	4.253	5.236	5.313	4.245	2.433
Min $\Delta \log \widetilde{P}(t_i)$ (1.0e−004)	−0.17	−0.15	−0.14	−0.12	−0.15	−0.17	−0.18
Max $\Delta \log \widetilde{P}(t_i)$ (1.0e−004)	0.18	0.16	0.14	0.12	0.17	0.13	0.16
Ave $\Delta \log \widetilde{P}(t_i)$ (1.0e−004)	0.0105	0.0030	0.0019	−0.0018	−0.0013	−0.0020	−0.0033
Std $\Delta \log \widetilde{P}(t_i)$ (1.0e−004)	0.0278	0.0197	0.0168	0.0165	0.0168	0.0172	0.0175
Min $\widetilde{P}(t_i)$	108.76	108.08	108.18	107.44	106.94	107.50	106.58
Max $\widetilde{P}(t_i)$	116.27	116.47	116.63	116.94	116.76	116.93	117.34
Ave $\widetilde{P}(t_i)$	112.11	112.21	112.32	112.45	112.38	112.32	112.30
Std $\widetilde{P}(t_i)$	1.002	1.001	1.001	1.001	1.001	1.001	1.002

[a] The table contains summary statistics for the number of trades, durations in seconds, log returns in basis points, and prices in dollars for IBM stock during the whole month of August 2007. The total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and from transactions on the New York Stock Exchange (NYSE). All trades before 9:30 am or after 4 pm and the first trade after 9:30 am are discarded due to NYSE trading hours and mechanisms. Each trading hour column, for example 11–12, includes information about trades after 11 am (inclusive) and before 12 pm (exclusive). All trades that have multiple prices at the same time are counted once, and the averaged price over the multiple trades is used.

Using $(k-1)$'s selected according to the aforementioned rule, we subsample every kth observation, estimate the noise variances using Eq. (16), and report its summary statistics in the upper panel of Table 7. Results indicate that $\hat{q}$'s are greater in the opening hours such as 9:30–10:00, though the magnitudes are similar in other hours. We also calculate $\hat{\xi}_n$ according to Eq. (14) and present its summary statistics in the lower panel of Table 7. Overall results suggest that models without jumps in equilibrium prices are rejected for IBM equity markets.

Note that our overall sample period is 1 month and includes multiple fixed time intervals of 1 h or 30 min. We split usual daily NYSE trading hours from 9:30 am to 4:00 pm into 7 different fixed time intervals and apply our test multiple times for the presence of jumps in each fixed time interval according to our test statistic $\hat{\xi}_n$. Since the total number of trading days in August 2007 was 23, we apply our test 161 times. Because of this large number of tests, it is desirable to make a multiple testing adjustment to control for the overall significance level of our analysis. In this paper, we apply the step-down procedure for this purpose, as follows.[16]

In general multiple hypothesis testing, it can be assumed that there are h fixed intervals with length T during our sample period, and we perform tests to determine the presence of jumps in each fixed interval. The null hypotheses for these multiple tests can be written as $H_1, H_2, \ldots$, and H_h. In our analysis, these null hypotheses are the same as those set in Eq. (1). We compute realized test statistics $\hat{\xi}_{1,n}, \hat{\xi}_{2,n}, \ldots, \hat{\xi}_{h,n}$ and their associated p-values $p_1, p_2, \ldots, p_h$ based on the standard Gumbel distribution, as stated in Theorem 1. Then, we sort the associated p-values and let $O(1), O(2), \ldots, O(h)$ be the indices of the ordered p-values, such that $p_{O(1)} \leq p_{O(2)} \leq \cdots \leq p_{O(h)}$. If the overall error rate for our equilibrium jump tests is α', we can reject all hypotheses $H_{O(h^*)}$ whose multiple-test adjusted p-values $\widetilde{p_{O(h^*)}}$ satisfy the condition $\widetilde{p_{O(h^*)}} = \max_{j \leq h^*}\{(h-j+1)p_{O(j)}\} \leq \alpha'$. We used this approach to estimate jump intensity. In particular, we count during how many time intervals we reject the individual null hypotheses using this

[16] We thank a referee for suggesting this approach in this context. Alternatives to the step-down procedure for controlling the overall error rate are the Bonferroni adjustment or incorporating False Discovery Rates.

Table 7
Empirical evidence on IBM stock trades during August 2007.[a]

Trading hour	9:30–10	10–11	11–12	12–13	13–14	14–15	15–16
Min $\hat{q}$ (%)	0.0109	0.0112	0.0091	0.0090	0.0104	0.0075	0.0069
Max $\hat{q}$ (%)	0.0649	0.0291	0.0244	0.0269	0.0244	0.0277	0.0287
Ave $\hat{q}$ (%)	0.0274	0.0183	0.0156	0.0145	0.0152	0.0152	0.0154
Std $\hat{q}$ (%)	0.0116	0.0054	0.0045	0.0042	0.0040	0.0051	0.0058
Min $\hat{\xi}_n$	−9.2903	−0.6920	−2.2385	−1.4288	−2.3117	−2.8200	1.7079
Max $\hat{\xi}_n$	−0.4648	7.5579	6.3403	15.1187	8.1149	9.6373	15.6170
Ave $\hat{\xi}_n$	−5.2954	2.8050	2.1490	2.4305	1.7093	2.0877	6.4348
Std $\hat{\xi}_n$	1.9911	2.3601	2.5421	3.7502	2.6460	2.6626	4.1961

[a] The table contains summary statistics for estimated market quality parameter q, which is the dispersion measure of market microstructure noise, as in Eq. (5), and estimated Gumbel variables ξ_n, as in Eq. (14). We use IBM stock data during August 2007, and the total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and from transactions on the New York Stock Exchange (NYSE). All trades before 9:30 am or after 4 pm and the first trade after 9:30 am are discarded due to NYSE trading hours and mechanisms. Each trading hour column, for example 11–12, includes information about trades after 11 am (inclusive) and before 12 pm (exclusive). All trades that have multiple prices at the same time are counted once, and the averaged price over the multiple trades is used.

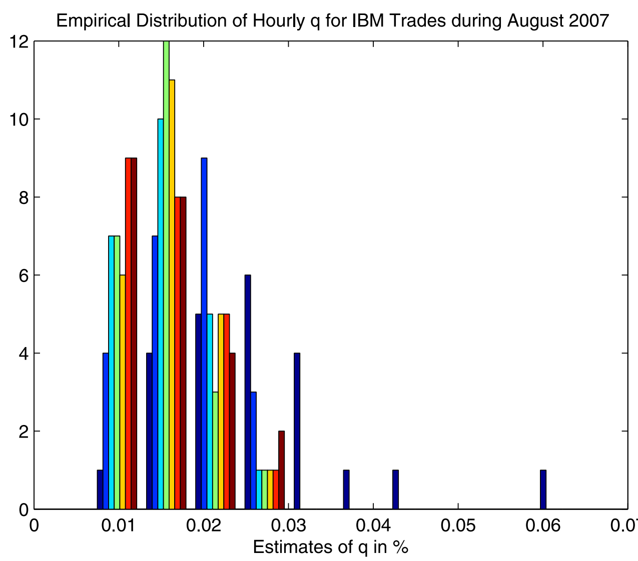

Fig. 2. Empirical distribution of hourly q for IBM trades during August 2007. This figure includes histograms of hourly q estimates. For every horizon, we calculate the sample autocorrelation to determine lag k in Eq. (16), using tick-by-tick data. We use IBM stock data during August 2007, and the total number of tick-by-tick observations used is 167,595. Data are collected from the TAQ database and from transactions on the New York Stock Exchange (NYSE). All trades before 9:30 am or after 4 pm and the first trade after 9:30 am are discarded due to NYSE trading hours and mechanisms. Different colors for each bin indicate different trading hours. Dark blue, regular blue, light blue, green, yellow, orange, and red represent trading hours 09:30–10, 10–11, 11–12, 12–13, 13–14, 14–15, and 15–16, respectively. Each trading hour, for example 11–12, includes information about trades after 11 am (inclusive) and before 12 pm (exclusive). Trades that have multiple prices at the same time are counted once, and the averaged price over the multiple trades is used for this analysis.

adjusted p-values. We find that with a 1% (5%) overall error rate, the efficient price jump intensity for IBM stocks is estimated at 3.11% (7.45%) during our sample period.

In Fig. 2, we graph the empirical distribution of IBM noise variance estimate $\hat{q}$'s. For each trading day, we have 7 different time horizons, and we obtain the noise variance by separately estimating the quantities over different time horizons. Different colors for each bin in Fig. 2 indicate different trading hours. Specifically, dark blue, regular blue, light blue, green, yellow, orange, and red represent trading hours 09:30–10, 10–11, 11–12, 12–13, 13–14, 14–15, and 15–16, respectively. As also reported in Table 7, estimates of $\hat{q}$ are centered around 0.01%–0.02%. Fig. 2 graphically shows that the noise level tends to be greater in the 9:30–10 interval (the dark blue bars) than in the other trading hours.

Finally, in Fig. 3, we compare graphically the asymptotic distribution of ξ and the empirical distribution of $\hat{\xi}_n$ based on our data. The asymptotic distribution is graphed with simulated data under the null hypothesis of no jump in equilibrium prices according to Eq. (14) in Theorem 1. The left panel in Fig. 3 includes the histogram of simulated ξ, which we would expect to see from data when there is no jump in equilibrium prices. The number of simulations is 6000. The right panel includes the histogram of $\hat{\xi}_n$ using our sample. As can be seen, we have different ranges in the distribution, which indicates the rejection of models with no jump in equilibrium prices. Therefore, we can conclude from this case study that models with jumps in efficient prices can better capture the intra-day dynamics of IBM stock price behavior.

7. Conclusion

Despite the empirical evidence of jumps documented in the asset pricing literature and the popularity of jump diffusion models to accommodate such evidence, the empirical market microstructure literature often ignores their presence in studies using high frequency data. This may be due to the difficulty of distinguishing between two unobservable components of observable data: noise and jumps in efficient prices. In this paper, we contribute to the literature by proposing new empirical methods which allow us to find evidence of jumps in underlying efficient price processes. These methods are immune to the presence of general noise and offer new empirical evidence. The approach suggested in this study is expected to be useful in various contexts such as event studies and arbitrage trading strategies as well as portfolio and risk management, among others.

Since we designed this test to take full advantage of ultra high frequency price data, the test can be applied to all sorts of price level data for local averaging as long as high frequency observations are available, so that our asymptotic arguments with a large number of observations in fixed time intervals are valid in the application. We suggest nonparametric methods, which would give evidence robust to model specification. It is important to note that we can investigate the equilibrium price jumps in the presence of general dependent noise processes, which is a crucial feature of noise patterns in financial markets. This general assumption on dependence in noise processes distinguishes our test from existing jump tests.

Through a simulation study, we show that our test has reasonable finite sample properties as long as block size for preprocessing is appropriately chosen. Our empirical study using IBM stock trades on the New York Stock Exchange indicates that there is a strong need to incorporate the presence of jumps in underlying pricing models.

Finally, in this paper, we only study how to identify jumps with finite activity in the presence of noise using high frequency data. It would be interesting to investigate the case of Lévy type jumps with infinite activity to determine whether evidence of Lévy jumps is due to the presence of noise or not.

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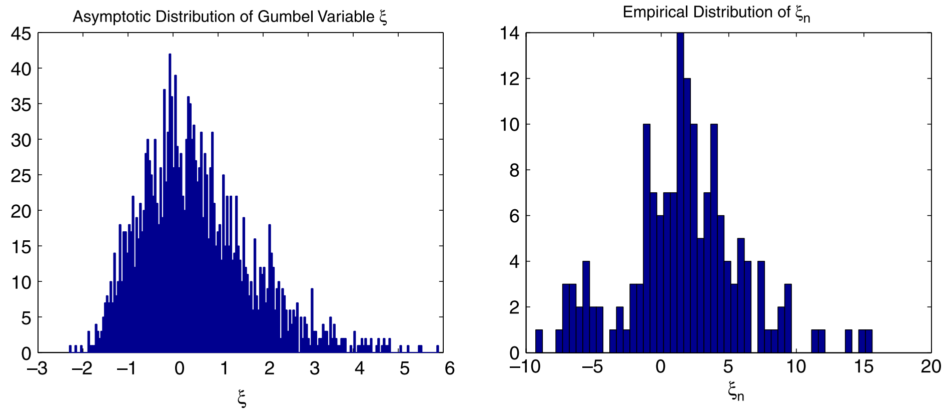

Fig. 3. Comparison of limiting and empirical distributions. The left panel includes the histogram of simulated Gumbel variables ξ, which is expected under the null hypothesis of no jumps in equilibrium prices. The number of simulations is 6000. The right panel includes the histogram of $\hat{\xi}_n$ for IBM stock trades during August 2007. The summary statistics for the distribution of $\hat{\xi}_n$ are listed in Table 7. The total number of tick-by-tick observations used is 167,595. Descriptions of data such as time horizon T used, number of observations n during $[0, T]$, durations Δt_i over each hour, and price $\tilde{P}(t_i)$ and log return $\Delta \log \tilde{P}(t_i)$ levels are reported in Table 6. Data are collected from the TAQ database and from transactions on the New York Stock Exchange (NYSE). All trades before 9:30 am or after 4 pm and the first trade after 9:30 am are discarded due to NYSE trading hours and mechanisms. Trades that have multiple prices at the same time are counted once, and the averaged price over the multiple trades is used.

Appendix

A.1. The nonzero drift

Our theoretical results are not affected by the nonzero drift. We provide a modified version of $\mathcal{X}(t_j)$ with the nonzero drift as follows: for any given $t_j \in \mathcal{G}_n^{kM}$, we can define a modified test by $\widetilde{\mathcal{X}}(t_j) = \frac{\sqrt{M}}{\sqrt{V_n}}(\mathcal{L}(t_j) - \hat{\mu}_{\mathcal{L}})$, where $\hat{\mu}_{\mathcal{L}} = \frac{1}{\lfloor \frac{n}{kM} \rfloor} \sum_{t_j \in \mathcal{G}_n^{kM}} \mathcal{L}(t_j)$ and V_n is estimated as discussed in Section 4.2. This definition demeans $\mathcal{L}(t_j)$ at time $t_j \in \mathcal{G}_n^{kM}$ using its averaged value, assuming that the nonzero drift is constant. We note that $\hat{\mu}_{\mathcal{L}} = O_p((\frac{n}{kM})^{-1/2})$, as $n \to \infty$. In the presence of the Poisson-type jumps we consider in this paper, the impact of jumps on the drift estimation becomes negligible as $n \to \infty$, because of the property of Poisson processes that there can be only finitely many jumps.

A.2. Proof of Theorem 1

We use the following Lemma 2, the proof of which can be found in Berman (1964) under the general asymptotic mixing condition as stated. The result is also mentioned in Ljung (1993).

Lemma 2. *Let $\mathcal{Z}(j)$ be a stationary Gaussian process, so that $E\mathcal{Z}(j) = 0$ and $E\mathcal{Z}^2(j) = 1$ for all $j = 0, 1, \ldots, n$. Furthermore, its covariance sequence $\rho_k = E\mathcal{Z}(0)\mathcal{Z}(k)$ with $\sum_{k=1}^{\infty} \rho_k^2 < \infty$, or $\lim_{k\to\infty} \rho_k \log k = 0$. Then, as $n \to \infty$,*

$$\frac{\max_j |\mathcal{Z}(j)| - A_n}{B_n} \xrightarrow{\mathcal{D}} \xi, \tag{20}$$

where ξ follows a standard Gumbel distribution, A_n and B_n are $A_n = (2\log n)^{1/2} - \frac{\log \pi + \log(\log n)}{2(2\log n)^{1/2}}$ and $B_n = \frac{1}{(2\log n)^{1/2}}$.

We apply Lemma 2 to our situation in Theorem 1. Notice that for all $t_j \in \mathcal{G}_n^{kM}$, $\mathcal{L}(t_j) = \widehat{P}(t_{j+kM}) - \widehat{P}(t_j) = \frac{1}{M}\sum_{i=\lfloor j/k \rfloor}^{\lfloor j/k \rfloor + M-1} [P(t_{(i+M)k}) - P(t_{ik})] + \frac{1}{M}\sum_{i=\lfloor j/k \rfloor}^{\lfloor j/k \rfloor + M-1} [U(t_{(i+M)k}) - U(t_{ik})]$. For the signal averages, we have approximation of $P(t_{(i+M)k}) - P(t_{ik})$ by a normal random variable. For the noise averages, we note from Theorem 3 in Mykland and Zhang (2011) that $\frac{1}{M}\sum_{i=\lfloor j/k \rfloor}^{\lfloor j/k \rfloor + M-1} [U(t_{(i+M)k}) - U(t_{ik})]$ satisfies the extra condition (*i.e.,* Gaussianity) under a contiguous measure P_n. The likelihood ratio dP/dP_n can asymptotically be expressed in terms of a polynomial sum of terms of the form $\sum_{i=\lfloor j/k \rfloor}^{\lfloor j/k \rfloor + M-1} U(t_{ik})$. The maximum of $|\mathcal{X}(t_j)|$ is taken over $t_j \in \mathcal{G}_n^{kM}$ and (under P_n) asymptotically independent of dP/dP_n. This proves Theorem 1.

A.3. Proof of Theorem 2

Under the alternative hypothesis when the jump times are $\tau_f \in [0, T]$, we have

$$\begin{aligned}
\max_{t_j \in \mathcal{G}_n^{kM}} \mathcal{L}(t_j)|_{alternative} &= \max_{t_j \in \mathcal{G}_n^{kM}} |\widehat{P}(t_{j+kM}) - \widehat{P}(t_j)| \\
&= \max_{t_j \in \mathcal{G}_n^{kM}} \left| \frac{1}{M} \sum_{i=\lfloor j/k \rfloor}^{\lfloor j/k \rfloor + M-1} \int_{t_{ik}}^{t_{(i+M)k}} Y(s) dJ(s) \right| + o_p(1) \\
&= \max_{t_j \in \mathcal{G}_n^{kM}, 1 \le f \le F} |Y(\tau_f)| \\
&\quad \times \frac{\text{Number of times } (t_{ik} \in (t_j, t_{j+kM-1}), t_{ik} \le \tau_f)}{M} \\
&= \max_{1 \le f \le F} |Y(\tau_f)| + o_p(1).
\end{aligned} \tag{21}$$

A.4. Proof of Proposition 1 under the null

Denote u_i and u as random variables with a mean of 0 and a variance of 1 along with n_i being a standard normal random variable. Using $P(t_m) - P(t_{m+k}) = \sigma\sqrt{k\Delta t}n_i$ and

$$U(t_m) - U(t_{m+k}) = q(u_m - u_{m+k}),$$

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} \widehat{Q}^2(k) &= \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n'} \sum_{m=1}^{n'} (\widetilde{P}(t_m) - \widetilde{P}(t_{m+k}))^2 \right) \\
&= \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n'} \sum_{m=1}^{n'} (P(t_m) - P(t_{m+k}) \right. \\
&\qquad \left. + U(t_m) - U(t_{m+k}))^2 \right) \\
&= \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n'} \sum_{m=1}^{n'} (\sigma\sqrt{k\Delta t}n_i + q(u_m - u_{m+k}))^2 \right) \\
&= q^2 E(u_m - u_{m+k})^2 = q^2 E(\sqrt{2}u)^2 = (\sqrt{2}q)^2.
\end{aligned}$$

A.5. Proof of Proposition 1 under the alternative

We assume there are F numbers of rare Poisson jumps in the efficient price process with F being finite over any fixed time horizon. The presence of jumps now comes into our efficient prices as $P(t_m) - P(t_{m+k}) = \sigma\sqrt{k\Delta t}n_m + O_p(1)I_{\tau\in[t_m,t_{m+k}]}$, where τ is the jump arrival time. Then,

$$\begin{aligned}
\operatorname*{plim}_{n\to\infty} \widehat{Q}^2(k) &= \operatorname*{plim}_{n\to\infty} \left(\frac{1}{n'} \sum_{m=1}^{n'} (\widetilde{P}(t_m) - \widetilde{P}(t_{m+k}))^2 \right) \\
&= \operatorname*{plim}_{n\to\infty} \frac{1}{n'} \sum_{m=1}^{n'} (O_p(1)I_\tau + q(u_m - u_{m+k})|)^2 \\
&= \underbrace{\operatorname*{plim}_{n\to\infty} \frac{1}{n'} \sum_{\text{with jump}} (O_p(1)I_\tau + q(u_m - u_{m+k}))^2}_{=\frac{F}{n'}O_p(1)\to 0 \text{ as } n\to\infty} \\
&\quad + \operatorname*{plim}_{n\to\infty} \frac{1}{n'} \sum_{\text{without jump}} (q(u_m - u_{m+k}))^2 \\
&= q^2 E((u_m - u_{m+k}))^2 = q^2 E(\sqrt{2}u)^2 = (\sqrt{2}q)^2.
\end{aligned}$$

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