Supplement to
"The Five Trolls under the Bridge: Principal Component Analysis with Asynchronous and Noisy High Frequency Data"
by Dachuan Chen, Per A. Mykland, and Lan Zhang
Journal of the American Statistical Association

Appendices

This supplement contains the proofs of the theorems and other mathematical results in the main body of the paper (Appendix A-F), as well as additional simulation results (Appendix G). References are to the main paper unless otherwise is indicated.

A Decomposition of the Smoothed TSRV Estimator

We only show the case when $r = s$ and $0 \leq t \leq \mathcal{T}$. The proof for other cases (i.e., $1 \leq r, s \leq d$) will be similar. Recall the definition of the S-TSRV as follows:

$$\widehat{\langle X, X \rangle}_t = \frac{1}{(1 - b/N)(K - J)} \left\{ K \widetilde{\left[\bar{Y}, \bar{Y}\right]}_t^{(K)} - J \widetilde{\left[\bar{Y}, \bar{Y}\right]}_t^{(J)} \right\},$$

where for a pair (J, K), and $N^*(t)$ defined in (2.3), we set

$$K \widetilde{\left[\bar{Y}, \bar{Y}\right]}_t^{(K)} = \frac{1}{2} \sum_{i=1}^{J} \left(\bar{Y}_{i+K} - \bar{Y}_i\right)^2 + \sum_{i=J+1}^{N^*(t)-b} \left(\bar{Y}_{i+K} - \bar{Y}_i\right)^2 + \frac{1}{2} \sum_{i=N^*(t)-b+1}^{N^*(t)-K} \left(\bar{Y}_{i+K} - \bar{Y}_i\right)^2$$

with

$$b = K + J.$$

We define $J \widetilde{\left[\bar{Y}, \bar{Y}\right]}_t^{(J)}$ similarly by switching J and K.

Recall the results of Theorem 1, Proposition 1 and Theorem 3 in Mykland et al. (2019), if we assume that

$\Delta\tau_n^+ \to 0, M_n^- \to \infty$ and $K-J \to \infty$ as $n \to \infty$, we have the following expression:

$$\begin{aligned}
\widehat{\langle X, X\rangle}_t &= \underbrace{\frac{1}{K-J}\left[\frac{1}{2}\left(\sum_{i=1}^{N^*(t)-b} + \sum_{i=J+1}^{N^*(t)-K}\right)\left(X_{\tau_{i+K-1}} - X_{\tau_i}\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{N^*(t)-b} + \sum_{i=K+1}^{N^*(t)-J}\right)\left(X_{\tau_{i+J-1}} - X_{\tau_i}\right)^2\right]}_{\text{Signal Part}} \\
&\quad \underbrace{-\frac{2}{K-J}\left(\sum_{i=1}^{N^*(t)-K} - \frac{1}{2}\sum_{i=1}^{b-K} - \frac{1}{2}\sum_{i=N^*(t)-b+1}^{N^*(t)-K}\right)\bar{\epsilon}_i\bar{\epsilon}_{i+K} + \frac{2}{K-J}\left(\sum_{i=1}^{N^*(t)-J} - \frac{1}{2}\sum_{i=1}^{b-J} - \frac{1}{2}\sum_{i=N^*(t)-b+1}^{N^*(t)-J}\right)\bar{\epsilon}_i\bar{\epsilon}_{i+J}}_{\text{Noise U-Statistics}} \\
&\quad + O_p\left(\left(\Delta\tau_n^+ + \left(M_n^-\right)^{-1}\right)^{\frac{1}{2}}\right). \qquad \text{(A.1)}
\end{aligned}$$

A. 1 Edge Part of Noise U-Statistics

According to formula (A.1), we know that the main martingale part for the noise U-Statistics of the estimator $\widehat{\langle X, X\rangle}_t$ is:

$$-\frac{2}{K-J}\sum_{i=1}^{N^*(t)-K}\bar{\epsilon}_i\bar{\epsilon}_{i+K} + \frac{2}{K-J}\sum_{i=1}^{N^*(t)-J}\bar{\epsilon}_i\bar{\epsilon}_{i+J},$$

and its edge part is:

$$\begin{aligned}
&\frac{1}{K-J}\left(\sum_{i=1}^{b-K} + \sum_{i=N^*(t)-b+1}^{N^*(t)-K}\right)\bar{\epsilon}_i\bar{\epsilon}_{i+K} - \frac{1}{K-J}\left(\sum_{i=1}^{b-J} + \sum_{i=N^*(t)-b+1}^{N^*(t)-J}\right)\bar{\epsilon}_i\bar{\epsilon}_{i+J} \\
=\ &\frac{1}{K-J}\left(\sum_{i=K+1}^{b} + \sum_{i=N^*(t)-J+1}^{N^*(t)}\right)\bar{\epsilon}_{i-K}\bar{\epsilon}_i - \frac{1}{K-J}\left(\sum_{i=J+1}^{b} + \sum_{i=N^*(t)-K+1}^{N^*(t)}\right)\bar{\epsilon}_{i-J}\bar{\epsilon}_i \\
=\ &-\frac{1}{K-J}\sum_{i=J+1}^{K}\bar{\epsilon}_{i-J}\bar{\epsilon}_i - \frac{1}{K-J}\sum_{i=N^*(t)-K+1}^{N^*(t)-J}\bar{\epsilon}_{i-J}\bar{\epsilon}_i \\
&\underbrace{+\frac{1}{K-J}\sum_{i=K+1}^{b}\left(\bar{\epsilon}_{i-K} - \bar{\epsilon}_{i-J}\right)\bar{\epsilon}_i + \frac{1}{K-J}\sum_{i=N^*(t)-J+1}^{N^*(t)}\left(\bar{\epsilon}_{i-K} - \bar{\epsilon}_{i-J}\right)\bar{\epsilon}_i}_{\text{(I)}},
\end{aligned}$$

where

$$\text{(I)} = O_p\left(\frac{J^{1/2}}{(K-J)\,M_n^-}\right) = o_p\left(\left(M_n^-\right)^{-1}\right),$$

and

$$-\frac{1}{K-J}\sum_{i=J+1}^{K}\bar{\epsilon}_{i-J}\bar{\epsilon}_i - \frac{1}{K-J}\sum_{i=N^*(t)-K+1}^{N^*(t)-J}\bar{\epsilon}_{i-J}\bar{\epsilon}_i = O_p\left((K-J)^{-1/2}\left(M_n^-\right)^{-1}\right).$$

If we let

$$K - J = O_p\left(\left(N/M_n^-\right)^{2/3}\right),$$

then $(K-J)^{-1/2}\left(M_n^-\right)^{-1} = O_p\left(N^{-1/3}\left(M_n^-\right)^{-2/3}\right)$. Comparing to the order of the edge term discussed in Proposition 1 of Mykland et al. (2019), for example, of order $O_p\left(J^{1/2}\left(\Delta\tau_n^+ + \left(M_n^-\right)^{-1}\right)^{1/2}\left(\Delta\tau_n^+\right)^{1/2}\right) = O_p\left(N^{-1} + N^{-1/2}\left(M_n^-\right)^{-1/2}\right)$, we know that

$$\begin{aligned}
\frac{N^{-1/3}\left(M_n^-\right)^{-2/3}}{N^{-1}} &= \left(\frac{N}{M_n^-}\right)^{\frac{2}{3}} \to \infty, \\
\frac{N^{-1/3}\left(M_n^-\right)^{-2/3}}{N^{-1/2}\left(M_n^-\right)^{-1/2}} &= \left(\frac{N}{M_n^-}\right)^{\frac{1}{6}} \to \infty.
\end{aligned}$$

Thus, we know that for the edge effect in noise U-statistics, the part that really matters for the AVAR estimator is

$$-\frac{1}{K-J}\sum_{i=J+1}^{K}\bar{\epsilon}_{i-J}\bar{\epsilon}_i - \frac{1}{K-J}\sum_{i=N^*(t)-K+1}^{N^*(t)-J}\bar{\epsilon}_{i-J}\bar{\epsilon}_i.$$

It is worth to mention that because the rate of convergence of the estimator is $O_p\left(N^{-1/6}\left(M_n^-\right)^{-1/3}\right)$, which is equivalent to the order $O_p\left(\left[(K-J)\,\Delta\tau_n^+\right]^{\frac{1}{2}}\right)$ under the Assumption 3. Then without loss of generality, denote $O_p\left(N^{-1/6}\left(M_n^-\right)^{-1/3}\right)$ by $O_p\left(a_n\right)$, then we have:

$$-\frac{1}{K-J}\sum_{i=J+1}^{K}\bar{\epsilon}_{i-J}\bar{\epsilon}_i - \frac{1}{K-J}\sum_{i=N^*(t)-K+1}^{N^*(t)-J}\bar{\epsilon}_{i-J}\bar{\epsilon}_i = O_p\left(a_n^2\right).$$

In the next section, we are going to find the edge term in the signal part which has the order $O_p\left(a_n^2\right)$.

A. 2 Further Decomposition of Signal Part

Based on the definition of the signal part in formula (A.1), we obtain for $\widehat{\langle X, X\rangle}_t$ that

$$
\begin{aligned}
& \frac{1}{2}\left(\sum_{i=1}^{N^*(t)-b}+\sum_{i=J+1}^{N^*(t)-K}\right)\left(X_{\tau_{i+K-1}}-X_{\tau_i}\right)^2-\frac{1}{2}\left(\sum_{i=1}^{N^*(t)-b}+\sum_{i=K+1}^{N^*(t)-J}\right)\left(X_{\tau_{i+J-1}}-X_{\tau_i}\right)^2 \\
= & \frac{1}{2}\left(2\sum_{i=1}^{N^*(t)-K}-\sum_{i=1}^{J}-\sum_{i=N^*(t)-b+1}^{N^*(t)-K}\right)\left(X_{\tau_{i+K-1}}-X_{\tau_i}\right)^2-\frac{1}{2}\left(2\sum_{i=1}^{N^*(t)-J}-\sum_{i=1}^{K}-\sum_{i=N^*(t)-b+1}^{N^*(t)-J}\right)\left(X_{\tau_{i+J-1}}-X_{\tau_i}\right)^2 \\
= & \underbrace{\sum_{i=J}^{N^*(t)-(K-J)-1}\left(X_{\tau_{i+K-J}}-X_{\tau_i}\right)^2}_{\text{(Sum of Squared Terms)}}+\underbrace{\frac{1}{2}\sum_{i=J+1}^{K}\left(X_{\tau_{i+J-1}}-X_{\tau_i}\right)^2-\frac{1}{2}\sum_{i=J}^{2J-1}\left(X_{\tau_{i+K-J}}-X_{\tau_i}\right)^2}_{\text{(II)}} \\
& \underbrace{-\frac{1}{2}\sum_{i=N^*(t)-K}^{N^*(t)-(K-J)-1}\left(X_{\tau_{i+K-J}}-X_{\tau_i}\right)^2-\frac{1}{2}\sum_{i=N^*(t)-K+1}^{N^*(t)-J}\left(X_{\tau_{i+J-1}}-X_{\tau_i}\right)^2}_{\text{(III)}} \\
& +\underbrace{\left(\sum_{i=1}^{N^*(t)-b}+\sum_{i=J+1}^{N^*(t)-K}\right)\left(X_{\tau_{i+K-1}}-X_{\tau_{i+J-1}}\right)\left(X_{\tau_{i+J-1}}-X_{\tau_i}\right)}_{\text{(IV)}},
\end{aligned}
$$

where

$$
\begin{aligned}
\text{(II)}+\text{(III)} &= O_p\left(J\left(K-J\right)\Delta\tau_n^+\right), \\
\text{(IV)} &= O_p\left(J\left(K-J\right)\Delta\tau_n^+\right).
\end{aligned}
$$

Moreover, the main part of the squared terms can be decomposed as follows:

$$
\sum_{i=J}^{N^*(t)-1-(K-J)}\left(X_{i+(K-J)}-X_i\right)^2=R_t^{X_{(2)}}+R_t^{X_{(1,1)}}-C_t^{X_{(2)}}-C_t^{X_{(1,1)}}, \tag{A.2}
$$

where

$$
\begin{aligned}
R_t^{X_{(2)}} &= (K-J)\sum_{i=J+1}^{N^*(t)}\Delta X_{\tau_i}^2, \\
R_t^{X_{(1,1)}} &= 2\sum_{p=1}^{K-J-1}(K-J-p)\sum_{i=J+p+1}^{N^*(t)}\Delta X_{\tau_{i-p}}\Delta X_{\tau_i},
\end{aligned}
$$

and

$$
\begin{aligned}
C_t^{X_{(2)}} &= \sum_{i=1}^{K-J-1} (K-J-i)\,\Delta X_{\tau_{J+i}}^2 + \sum_{i=0}^{K-J-1} (K-J-i)\,\Delta X_{\tau_{N^*(t)-i}}^2, \\
C_t^{X_{(1,1)}} &= 2\sum_{p=1}^{K-J-1}\sum_{i=1}^{K-J-p} (K-J-p-i)\,\Delta X_{\tau_{J+i}}\Delta X_{\tau_{J+i+p}} + 2\sum_{p=1}^{K-J-1}\sum_{i=0}^{K-J-p} (K-J-p-i)\,\Delta X_{\tau_{N^*(t)-i-p}}\Delta X_{\tau_{N^*(t)-i}}.
\end{aligned}
$$

Observe that $C_t^{X_{(2)}} = O_p\left((K-J)^2\,\Delta\tau_n^+\right)$ and $C_t^{X_{(1,1)}} = O_p\left((K-J)^2\,\Delta\tau_n^+\right)$.

If we let $K-J = O_p\left((N/M_n^-)^{2/3}\right)$, then based on all of above calculations, we obtain:

$$
\begin{aligned}
\text{Signal Part in formula (A.1)} &= \sum_{i=J+1}^{N^*(t)} \Delta X_{\tau_i}^2 + 2\sum_{p=1}^{K-J-1}\left(\frac{K-J-p}{K-J}\right)\sum_{i=J+p+1}^{N^*(t)} \Delta X_{\tau_{i-p}}\Delta X_{\tau_i} \\
&\quad -\frac{1}{K-J}\left(C_t^{X_{(2)}} + C_t^{X_{(1,1)}}\right) + o_p\left(a_n^2\right).
\end{aligned}
$$

B Proof of Lemma 1

Based on formulae (2.7), (3.2) and (3.3), the estimation error of $\hat{c}_{\Delta T_n,t}^{(r,s)}$ can be separated into two parts:

$$
\begin{aligned}
\bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)} &= \frac{1}{\Delta T_n}\int_t^{t+\Delta T_n} (t+\Delta T_n - u)\,dc_u^{(r_1,s_1)}, \\
\hat{c}_{\Delta T_n,t}^{(r,s)} - \bar{c}_{\Delta T_n,t}^{(r,s)} &= \frac{1}{\Delta T_n}\left(M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right) + \frac{1}{\Delta T_n}\left(\tilde{e}_{t+\Delta T_n}^{(r,s)} - \tilde{e}_t^{(r,s)}\right).
\end{aligned} \tag{B.1}
$$

By Lemma 1 and 4 of Mykland and Zhang (2006), we know that $\left\|\bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}\right\|_2^2 = O_p\left(\Delta T_n\right)$, then for $\varepsilon > 0$,

$$
\sup_t \left|\bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}\right| = O_p\left(\Delta T_n^{1/2-\varepsilon}\right).
$$

Because $\Delta\tau_n^+ = o_p\left(a_n^2\right)$ and $a_n^2 = o_p\left(\Delta T_n\right)$, we have:

$$
\begin{aligned}
&\frac{1}{\Delta T_n}\int_t^{t+\Delta T_n} (t+\Delta T_n - u)\,dc_u^{(r_1,s_1)} \\
&= \bar{\beta}_{\Delta T_n,t}^{(r,s)} + \frac{1}{\Delta T_n}\int_{\tau_{N^*(t+\Delta T_n)}}^{t+\Delta T_n} (t+\Delta T_n - u)\,dc_u^{(r_1,s_1)} - \frac{1}{\Delta T_n}\int_{\tau_{N^*(t)}}^{t} (t+\Delta T_n - u)\,dc_u^{(r_1,s_1)} \\
&= \bar{\beta}_{\Delta T_n,t}^{(r,s)} + O_p\left(\Delta T_n^{-1}\left(\Delta\tau_n^+\right)^{3/2-\varepsilon}\right) + O_p\left(\left(\Delta\tau_n^+\right)^{1/2-\varepsilon}\right) \\
&= \bar{\beta}_{\Delta T_n,t}^{(r,s)} + o_p\left((K-J)^{-3/2+\varepsilon}\,\Delta T_n^{-1}\left(\Delta T_n\right)^{3/2-\varepsilon}\right) + o_p\left((K-J)^{-1/2+\varepsilon}\left(\Delta T_n\right)^{1/2-\varepsilon}\right) \\
&= \bar{\beta}_{\Delta T_n,t}^{(r,s)} + o_p\left(\Delta T_n^{1/2}\right).
\end{aligned}
$$

Moreover, by Definition (2.8), we have $\frac{1}{\Delta T_n}\left(M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right) = \tilde{\beta}_{\Delta T_n,t}^{(r,s)}$ and $\left\|M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right\|_2^2 = O_p\left(a_n^2 \Delta T_n\right)$. Thus,

$$\sup_t \left|\frac{1}{\Delta T_n}\left(M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right)\right| = O_p\left(\Delta T_n^{-1}\left(a_n^2 \Delta T_n\right)^{1/2-\varepsilon}\right).$$

Finally, we have $\sup_t \left|\frac{1}{\Delta T_n}\left(\tilde{e}_{t+\Delta T_n}^{(r,s)} - \tilde{e}_t^{(r,s)}\right)\right| = O_p\left(\Delta T_n^{-1}\left(a_n^4\right)^{1/2-\varepsilon}\right) = o_p\left(\Delta T_n^{-1}\left(a_n^2 \Delta T_n\right)^{1/2}\right)$. Thus the asymptotic representation of the estimation error is as follows:

$$\begin{aligned} \sup_t \left|\bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}\right| &= O_p\left(\Delta T_n^{1/2-\varepsilon}\right), \\ \sup_t \left|\hat{c}_{\Delta T_n,t}^{(r,s)} - \bar{c}_{\Delta T_n,t}^{(r,s)}\right| &= O_p\left(\Delta T_n^{-1}\left(a_n^2 \Delta T_n\right)^{1/2-\varepsilon}\right). \end{aligned} \tag{B.2}$$

By (B.2), we have:

$$\sup_t \left|\hat{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}\right| = O_p\left(\Delta T_n^{1/2-\varepsilon}\right) + O_p\left(\Delta T_n^{-1}\left(a_n^2 \Delta T_n\right)^{1/2-\varepsilon}\right),$$

and it is obvious that if ΔT_n satisfies (3.1), then

$$\sup_t \left|\hat{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}\right| = o_p(1).$$

C Proof of Lemma 2

Recall the formulas (2.7), (3.2) and (3.3), the estimation error of $\hat{c}_{\Delta T_n,t}^{(r,s)}$ can be expressed as:

$$\hat{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)} = \underbrace{\bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)}}_{O_p\left(\Delta T_n^{1/2}\right)} + \frac{1}{\Delta T_n}\left(M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right) + \underbrace{\frac{1}{\Delta T_n}\left(\tilde{e}_{t+\Delta T_n}^{(r,s)} - \tilde{e}_t^{(r,s)}\right)}_{O_p\left(a_n^2 \Delta T_n^{-1}\right)}. \tag{C.1}$$

Recall that $\frac{1}{\Delta T_n}\left(M_{t+\Delta T_n}^{(r,s)} - M_t^{(r,s)}\right) = \tilde{\beta}_{\Delta T_n,t}^{(r,s)}$, and by definition (3.3), we know that

$$\beta_{\Delta T_n,t}^{(r,s)} = \tilde{\beta}_{\Delta T_n,t}^{(r,s)} + \bar{c}_{\Delta T_n,t}^{(r,s)} - c_t^{(r,s)} + \frac{1}{\Delta T_n}\left(\tilde{e}_{t+\Delta T_n}^{(r,s)} - \tilde{e}_t^{(r,s)}\right), \tag{C.2}$$

and thus

$$E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right) = E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)}\tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right) + O_p\left(\Delta T_n\right) + O_p\left(a_n^4 \Delta T_n^{-2}\right),$$

uniformly with respect to t. Therefore, if $a_n^{-1}\Delta T_n \to 0$ as $n \to \infty$, we have $\Delta T_n = o_p(a_n)$ and then

$$E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right) = E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)}\tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right) + O_p\left(a_n^4\Delta T_n^{-2}\right) + o_p(a_n). \tag{C.3}$$

Recall the decomposition (C.2), and by the Cauchy-Swartz inequality, we have:

$$\left\|\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)} - \tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)}\tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)}\right\|_2 = O_p\left(\Delta T_n^{-3/2}a_n^3\right),$$

and when $a_n^{-1}\Delta T_n \to 0$ as $n \to \infty$, $\Delta T_n^{-3/2}a_n^3 = o_p\left(\Delta T_n^{-1}a_n^2\right)$, thus,

$$\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)} = \tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)}\tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} + o_p\left(\Delta T_n^{-1}a_n^2\right), \tag{C.4}$$

uniformly with respect to t.

By the Minkowski inequality, we have

$$\begin{aligned} & \left\|\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)} - \frac{1}{\Delta T_n^2}\varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)}\right\|_2 \\ \leq \; & \left\|\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)} - E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right)\right\|_2 + \left\|E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right) - \frac{1}{\Delta T_n^2}\varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)}\right\|_2. \end{aligned}$$

C. 1 Bound of $\left\|\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)} - E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)}\beta_{\Delta T_n,t}^{(r_2,s_2)}|\mathcal{F}_t\right)\right\|_2$

For the simplicity of discussion, set $B_i^{(r,s)} = \tilde{B}_i^{(r,s)}[2]$, then

$$\tilde{\beta}_{\Delta T_n,t}^{(r,s)} = \frac{1}{\Delta T_n}\sum_{i=N^*(t)+1}^{N^*(t+\Delta T_n)} B_i^{(r,s)}, \tag{C.5}$$

and

$$\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)}\tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} = \frac{1}{\Delta T_n^2}\sum_{i=N^*(t)+1}^{N^*(t+\Delta T_n)} B_{t+\Delta T_n,i}^{(r_1,s_1)}B_{t+\Delta T_n,i}^{(r_2,s_2)} + \frac{1}{\Delta T_n^2}\sum_{i=N^*(t)+2}^{N^*(t+\Delta T_n)}\left(\sum_{l=1}^{i-N^*(t)-1} B_{t+\Delta T_n,i-l}^{(r_1,s_1)}\right)B_{t+\Delta T_n,i}^{(r_2,s_2)}[2], \tag{C.6}$$

where [2] denotes the summation by switching (r_1, s_1) and (r_2, s_2).

Assume $\gamma \in (\alpha, 2\alpha)$. Recall the decomposition (C.2), (C.4), (C.5) and (C.6), we know that

$$\left\| \beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} - E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} | \mathcal{F}_t \right) \right\|_2$$

has the same order as:

$$\pi_1 = \frac{1}{\Delta T_n^2} \sum_{i=N^*(t)+2}^{N^*(t+\Delta T_n)} \left(\sum_{l=1}^{i-N^*(t)-1} B_{t+\Delta T_n,i-l}^{(r_1,s_1)} \right) B_{t+\Delta T_n,i}^{(r_2,s_2)}[2].$$

In what follows, we prove $\|\pi_1\|_2 = O_p\left(a_n^2 \Delta T_n^{-1}\right)$. Note that

$$\begin{aligned} E\left[\pi_1^2 | \mathcal{F}_t\right] &= \frac{1}{\Delta T_n^4} E\left[\left(\sum_{i=N^*(t)+2}^{N^*(t+\Delta T_n)} \left(\sum_{l=1}^{i-N^*(t)-1} B_{t+\Delta T_n,i-l}^{(r_1,s_1)} \right) B_{t+\Delta T_n,i}^{(r_2,s_2)}[2] \right)^2 | \mathcal{F}_t \right] \\ &= \frac{1}{\Delta T_n^4} \sum_{i=N^*(t)+2}^{N^*(t+\Delta T_n)} E\left[\left(\sum_{l=1}^{i-N^*(t)-1} B_{t+\Delta T_n,i-l}^{(r_1,s_1)} \right)^2 \left(B_{t+\Delta T_n,i}^{(r_2,s_2)} \right)^2 [2] | \mathcal{F}_t \right] \\ &= \frac{1}{\Delta T_n^4} \sum_{i=N^*(t)+2}^{N^*(t+\Delta T_n)} \sum_{l=1}^{i-N^*(t)-1} E\left[\left(B_{t+\Delta T_n,i-l}^{(r_1,s_1)} \right)^2 \left(B_{t+\Delta T_n,i}^{(r_2,s_2)} \right)^2 | \mathcal{F}_t \right] [2], \end{aligned} \tag{C.7}$$

where

$$E\left[\left(B_{t+\Delta T_n,i-l}^{(r_1,s_1)} \right)^2 \left(B_{t+\Delta T_n,i}^{(r_2,s_2)} \right)^2 | \mathcal{F}_t \right] = O_p\left(\left((K-J)\left(\Delta \tau_n^+\right)^2 + \frac{1}{(K-J)^2 \left(M_n^-\right)^2} \right)^2 \right). \tag{C.8}$$

Substituting (C.8) into (C.7), we obtain:

$$E\left[\pi_1^2 | \mathcal{F}_t\right] = O_p\left(\frac{1}{\Delta T_n^4} \left(N^*(t+\Delta T_n) - N^*(t)\right)^2 \left((K-J)\left(\Delta \tau_n^+\right)^2 + \frac{1}{(K-J)^2 \left(M_n^-\right)^2} \right)^2 \right),$$

and if we make Assumption 3, then

$$\begin{aligned} & \frac{1}{\Delta T_n^4} \left(N^*(t+\Delta T_n) - N^*(t)\right)^2 \left((K-J)\left(\Delta \tau_n^+\right)^2 + \frac{1}{(K-J)^2 \left(M_n^-\right)^2} \right)^2 \\ \sim \quad & \Delta T_n^{-2} \left((K-J)\Delta \tau_n^+ + \frac{N}{(K-J)^2 \left(M_n^-\right)^2} \right)^2 \\ \sim \quad & a_n^4 \Delta T_n^{-2}, \end{aligned}$$

and we have $E\left[\pi_1^2|\mathcal{F}_t\right] = O_p\left(a_n^4 \Delta T_n^{-2}\right)$ uniformly with respect to t. Finally we obtain:

$$\sup_t \left\| \beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} - E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) \right\|_2 = O_p\left(a_n^2 \Delta T_n^{-1}\right).$$

C. 2 Bound of $\left\| E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) - \frac{1}{\Delta T_n^2} \varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)} \right\|_2$

First find the conditional expectation of $\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)} \tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)}$ as follows:

$$E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)} \tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) = \frac{1}{\Delta T_n^2} \sum_{i=N^*(t)+1}^{N^*(t+\Delta T_n)} E\left(B_{t+\Delta T_n,i}^{(r_1,s_1)} B_{t+\Delta T_n,i}^{(r_2,s_2)} |\mathcal{F}_t\right),$$

where

$$E\left(B_{t+\Delta T_n,i}^{(r_1,s_1)} B_{t+\Delta T_n,i}^{(r_2,s_2)} |\mathcal{F}_t\right) = \left[\sum_{p=1}^{K-J-1} \left(\frac{K-J-p}{K-J}\right)^2 \int_{\tau_{i-p-1}}^{\tau_{i-p}} c_u^{(r_1,r_2)} du\right] \int_{\tau_{i-1}}^{\tau_i} c_u^{(s_1,s_2)} du[2][2] + \frac{2\varsigma^{(r_1,r_2)}\varsigma^{(s_1,s_2)}}{(K-J)^2 \left(M_n^-\right)^2}[2][2]. \tag{C.9}$$

Finally, by formula (C.3), it is easy to see that

$$\begin{aligned} & \left\| E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) - \frac{1}{\Delta T_n^2} \varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)} \right\|_2 \\ \leq \; & \left\| E\left(\beta_{\Delta T_n,t}^{(r_1,s_1)} \beta_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) - E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)} \tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) \right\|_2 + \left\| E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)} \tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) - \frac{1}{\Delta T_n^2} \varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)} \right\|_2 \\ = \; & O_p\left(a_n^4 \Delta T_n^{-2}\right) + o_p\left(a_n\right), \end{aligned}$$

uniformly with respect to t, because $\sup_t \left\| E\left(\tilde{\beta}_{\Delta T_n,t}^{(r_1,s_1)} \tilde{\beta}_{\Delta T_n,t}^{(r_2,s_2)} |\mathcal{F}_t\right) - \frac{1}{\Delta T_n^2} \varphi_{\Delta T_n,t}^{(r_1,r_2,s_1,s_2)} \right\|_2 = O_p\left(a_n^8 \Delta T_n^{-3}\right) = o_p\left(a_n\right)$ by comparing (C.3), (C.9) and (3.4).

D Proof of Theorem 1

The estimation error can be decomposed as follows: for $1 \leq m \leq d$,

$$\hat{V}\left(\Delta T_n, X; F_m\right) - \int_0^{\mathcal{T}} F_m\left(c_s\right) ds - a_n^2 \Delta T_n^{-1} \varphi_{\mathcal{T}}^{(m)} = R^{\text{Expansion}} + R^{\text{Spot-V}} + R^{\text{Bias}} - R^{\text{Discrete}}, \tag{D.1}$$

where R^{Discrete} is defined in (4.2), $R^{\text{Spot-V}}$ and $R^{\text{Spot-B}}$ is defined in (4.3), and

$$\begin{aligned} R^{\text{Expansion}} &= \Delta T_n \sum_{i=1}^{B} \left(F_m\left(\hat{c}_{\Delta T_n, T_{n,i-1}}\right) - F_m\left(c_{T_{n,i-1}}\right) - \sum_{r_1,s_1=1}^{d} \partial_{r_1 s_1} F_m\left(c_{T_{n,i-1}}\right) \beta^{(r_1,s_1)}_{\Delta T_n, T_{n,i-1}} \right. \\ &\quad \left. -\frac{1}{2} \sum_{r_1,s_1,r_2,s_2=1}^{d} \partial^2_{r_1 s_1, r_2 s_2} F_m\left(c_{T_{n,i-1}}\right) \beta^{(r_1,s_1)}_{\Delta T_n, T_{n,i-1}} \beta^{(r_2,s_2)}_{\Delta T_n, T_{n,i-1}} \right), \qquad \text{(D.2)} \\ R^{\text{Bias}} &= R^{\text{Spot-B}} - a_n^2 \Delta T_n^{-1} \varphi^{(m)}_{\mathcal{T}}. \end{aligned}$$

First of all, it is straightforward to see that

$$R^{\text{Discrete}} = O_p\left(\Delta T_n\right). \qquad \text{(D.3)}$$

Next, because the symmetric function f is C^3 on $\mathcal{D}\left(g_1, g_2, \ldots, g_r\right)$, we obtain:

$$R^{\text{Expansion}} = O_p\left(\Delta T_n \sum_{i=1}^{B} \left(\left\| \boldsymbol{\beta}_{\Delta T_n, T_{n,i-1}} \right\|^3 \right) \right).$$

By result (ii) of Lemma 2, we know that $\left\| \boldsymbol{\beta}_{\Delta T_n, T_{n,i-1}} \right\|^3 = O_p\left(a_n^3 \Delta T_n^{-3/2}\right)$ and consequently, when $a_n^{-2} \Delta T_n \to \infty$,

$$R^{\text{Expansion}} = O_p\left(a_n^3 \Delta T_n^{-1}\right) = o_p\left(a_n\right). \qquad \text{(D.4)}$$

Thirdly, by result (ii) of Lemma 2, it is easy to see that

$$R^{\text{Spot-V}} = O_p\left(a_n\right). \qquad \text{(D.5)}$$

Lastly, we calculate the order of R^{Bias}, which could be defined as:

$$R^{\text{Bias}} = R^{\text{Spot-B}} - a_n^2 \Delta T_n^{-1} \varphi^{(m)}_{\mathcal{T}} = R^{\text{Bias-I}} + R^{\text{Bias-II}} + R^{\text{Bias-III}}, \qquad \text{(D.6)}$$

with

$$\vartheta^{(r_1,r_2,s_1,s_2)}_{T_{i-1}} = R^{\text{Spot-B}} - a_n^2 \Delta T_n^{-1} \varphi^{(m)}_{n,B,\mathcal{T}} = \beta^{(r_1,s_1)}_{\Delta T_n, T_{i-1}} \beta^{(r_2,s_2)}_{\Delta T_n, T_{i-1}} - \tilde{\beta}^{(r_1,s_1)}_{\Delta T_n, T_{i-1}} \tilde{\beta}^{(r_2,s_2)}_{\Delta T_n, T_{i-1}},$$

and

$$\varphi^{(m)}_{n,B,\mathcal{T}} = a_n^{-2} \int_0^{\mathcal{T}} \frac{1}{2} \sum_{r_1,s_1,r_2,s_2=1}^{d} \partial^2_{r_1 s_1, r_2 s_2} F_m\left(c_u\right) d\left[M^{(r_1,s_1)}, M^{(r_2,s_2)}\right]^{(B)}_u,$$

and

$$
\begin{aligned}
R^{\text{Bias-I}} &= \Delta T_n \sum_{i=1}^{B} \left[\frac{1}{2} \sum_{r_1,s_1,r_2,s_2=1}^{d} \partial^2_{r_1 s_1, r_2 s_2} F_m \left(c_{T_{n,i-1}} \right) \left(\vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} - E\left(\vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} | \mathcal{F}_{T_{i-1}} \right) \right) \right], \\
R^{\text{Bias-II}} &= \Delta T_n \sum_{i=1}^{B} \left[\frac{1}{2} \sum_{r_1,s_1,r_2,s_2=1}^{d} \partial^2_{r_1 s_1, r_2 s_2} F_m \left(c_{T_{n,i-1}} \right) E\left(\vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} | \mathcal{F}_{T_{i-1}} \right) \right], \\
R^{\text{Bias-III}} &= a_n^2 \Delta T_n^{-1} \varphi_{n,B,\mathcal{T}}^{(m)} - a_n^2 \Delta T_n^{-1} \varphi_{\mathcal{T}}^{(m)}.
\end{aligned}
$$

By formula (3.7) of result (iii) in Lemma 2, we know that $\sup_i \left\| \vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} - E\left(\vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} | \mathcal{F}_{T_{i-1}} \right) \right\|_2 = O_p\left(a_n^2 \Delta T_n^{-1} \right)$, and because $a_n^{-2} \Delta T_n \to \infty$,

$$
\left\| R^{\text{Bias-I}} \right\|_2 = O_p\left(a_n^2 \Delta T_n^{-1/2} \right) = o_p\left(a_n \right). \tag{D.7}
$$

Then by formula (3.6) of result (iii) in Lemma 2, we know that $\sup_i \left\| E\left(\vartheta_{T_{i-1}}^{(r_1,r_2,s_1,s_2)} | \mathcal{F}_{T_{i-1}} \right) \right\|_2 = O_p\left(a_n^4 \Delta T_n^{-2} \right) + o_p\left(a_n \right)$, and therefore

$$
\left\| R^{\text{Bias-II}} \right\|_2 = O_p\left(a_n^4 \Delta T_n^{-2} \right) + o_p\left(a_n \right). \tag{D.8}
$$

Note that $a_n^{-2} \left[M^{(r_1,s_1)}, M^{(r_2,s_2)} \right]_u^{(B)} \xrightarrow{p} \text{ACOV}\left(M^{(r_1,s_1)}, M^{(r_2,s_2)} \right)_u$ for all u and $(r_1,s_1), (r_2,s_2)$, we obtain $\varphi_{n,B,\mathcal{T}}^{(m)} \xrightarrow{p} \varphi_{\mathcal{T}}^{(m)}$ and thus,

$$
R^{\text{Bias-III}} = o_p\left(a_n^2 \Delta T_n^{-1} \right). \tag{D.9}
$$

Finally, by substituting (D.3)-(D.9) into (D.1), we obtain:

$$
a_n^{-2} \Delta T_n \left(\hat{V}\left(\Delta T_n, X; F_m \right) - \int_0^{\mathcal{T}} F_m\left(c_s \right) ds \right) - \varphi_{\mathcal{T}}^{(m)} = o_p\left(1 \right).
$$

E Proof of Theorem 2

Before the proof, we introduce notations as follows:

$$
\begin{aligned}
\psi_i^{(r,s)} &= \bar{c}_{\Delta T_n/2,(i-1/2)\Delta T_n}^{(r,s)} - \bar{c}_{\Delta T_n/2,(i-1)\Delta T_n}^{(r,s)}, \\
\breve{\beta}_{\Delta T_n,t}^{(r,s)} &= \hat{c}_{\Delta T_n,t}^{(r,s)} - \bar{c}_{\Delta T_n,t}^{(r,s)},
\end{aligned} \tag{E.1}
$$

where $\bar{c}^{(r,s)}_{\Delta T_n,t}$ is defined in (3.3). We also define:

$$\bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}} = \frac{1}{4}\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,(i-1)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n/2,(i-1)\Delta T_n} + \frac{1}{4}\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,(i-1/2)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n/2,(i-1/2)\Delta T_n}. \tag{E.2}$$

Moreover, recall the definition (3.3), we have:

$$\breve{\beta}^{(r,s)}_{\Delta T_n,t} = \tilde{\beta}^{(r,s)}_{\Delta T_n,t} + \frac{1}{\Delta T_n}\left(\tilde{e}^{(r,s)}_{t+\Delta T_n} - \tilde{e}^{(r,s)}_t\right) = \frac{1}{\Delta T_n}\left(M^{(r,s)}_{t+\Delta T_n} - M^{(r,s)}_t\right) + \frac{1}{\Delta T_n}\left(\tilde{e}^{(r,s)}_{t+\Delta T_n} - \tilde{e}^{(r,s)}_t\right).$$

Note that the estimation error could be decomposed as follows: for $1 \le m \le d$,

$$\tilde{V}\left(\Delta T_n, X; F_m\right) - \int_0^{\mathcal{T}} F_m\left(c_s\right) ds = R^{\text{Expansion}} + R^{\text{Spot-V}} + R^{\text{Adjusted-Bias}} - R^{\text{Discrete}}, \tag{E.3}$$

where R^{Discrete} is defined in (4.2), $R^{\text{Spot-V}}$ is defined in (4.3), $R^{\text{Expansion}}$ is defined in (D.2) and

$$R^{\text{Adjusted-Bias}} = R^{\text{Adjusted-Bias-I}} + R^{\text{Adjusted-Bias-II}} + R^{\text{Adjusted-Bias-III}} + R^{\text{Adjusted-Bias-IV}}, \tag{E.4}$$

with

$$\begin{aligned}
R^{\text{Adjusted-Bias-I}} &= \Delta T_n \sum_{i=1}^{B}\left[\frac{1}{2}\sum_{r_1,s_1,r_2,s_2=1}^{d}\left[\partial^2_{r_1s_1,r_2s_2}F_m\left(c_{T_{n,i-1}}\right) - \partial^2_{r_1s_1,r_2s_2}F_m\left(\hat{c}_{\Delta T_n,T_{n,i-1}}\right)\right]\beta^{(r_1,s_1)}_{\Delta T_n,T_{n,i-1}}\beta^{(r_2,s_2)}_{\Delta T_n,T_{n,i-1}}\right],\\
R^{\text{Adjusted-Bias-II}} &= \Delta T_n \sum_{i=1}^{B}\left[\frac{1}{2}\sum_{r_1,s_1,r_2,s_2=1}^{d}\partial^2_{r_1s_1,r_2s_2}F_m\left(\hat{c}_{\Delta T_n,T_{n,i-1}}\right)\left(\beta^{(r_1,s_1)}_{\Delta T_n,T_{n,i-1}}\beta^{(r_2,s_2)}_{\Delta T_n,T_{n,i-1}} - \breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n}\right)\right],\\
R^{\text{Adjusted-Bias-III}} &= \Delta T_n \sum_{i=1}^{B}\left[\frac{1}{2}\sum_{r_1,s_1,r_2,s_2=1}^{d}\partial^2_{r_1s_1,r_2s_2}F_m\left(\hat{c}_{\Delta T_n,T_{n,i-1}}\right)\left[\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}} - E\left(\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}|\mathcal{F}_{T_{n,i-1}}\right)\right]\right],\\
R^{\text{Adjusted-Bias-IV}} &= \Delta T_n \sum_{i=1}^{B}\left[\frac{1}{2}\sum_{r_1,s_1,r_2,s_2=1}^{d}\partial^2_{r_1s_1,r_2s_2}F_m\left(\hat{c}_{\Delta T_n,T_{n,i-1}}\right)E\left(\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}|\mathcal{F}_{T_{n,i-1}}\right)\right],
\end{aligned}$$

where $\beta^{(r_1,s_1)}_{\Delta T_n,T_{n,i-1}}$ is defined in (3.3),

$$\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{i-1}} = \breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - \hat{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}},$$

and $\breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n}$ is defined in (E.1).

If we assume $a_n^{-1}\Delta T_n \to 0$ and $a_n^{-3/2}\Delta T_n \to \infty$ as $n \to \infty$, then following from the results (D.3)-(D.5) in the

proof of Theorem 1, we obtain:

$$\begin{aligned} R^{\text{Discrete}} &= O_p\left(\Delta T_n\right) = o_p\left(a_n\right), \\ R^{\text{Spot-V}} &= O_p\left(a_n\right), \\ R^{\text{Expansion}} &= O_p\left(a_n^3 \Delta T_n^{-1}\right) = o_p\left(a_n\right). \end{aligned} \tag{E.5}$$

For $R^{\text{Adjusted-Bias-I}}$, because the symmetric function f is C^3 on $\mathcal{D}\left(g_1, g_2, \ldots, g_r\right)$, then we know that $\partial^2_{r_1 s_1, r_2 s_2} F_m$ is in C^1, and thus,

$$\sup_i \left\|\partial^2_{r_1 s_1, r_2 s_2} F_m\left(c_{T_{n,i-1}}\right) - \partial^2_{r_1 s_1, r_2 s_2} F_m\left(\hat{c}_{\Delta T_n, T_{n,i-1}}\right)\right\|_2 = O_p\left(\sup_i \left\|\hat{c}_{\Delta T_n, T_{n,i-1}} - c_{T_{n,i-1}}\right\|_2\right) = O_p\left(a_n \Delta T_n^{-1/2}\right).$$

Recall the result (ii) of Lemma 2, we have $\sup_i \left\|\beta^{(r_1,s_1)}_{\Delta T_n, T_{n,i-1}} \beta^{(r_2,s_2)}_{\Delta T_n, T_{n,i-1}}\right\|_2 = O_p\left(a_n^2 \Delta T_n^{-1}\right)$, and therefore

$$\left\|R^{\text{Adjusted-Bias-I}}\right\|_2 = O_p\left(a_n^3 \Delta T_n^{-1}\right) = o_p\left(a_n\right). \tag{E.6}$$

For $R^{\text{Adjusted-Bias-II}}$, because

$$\begin{aligned} & \beta^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n} \beta^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - \breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n} \breve{\beta}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} \\ = \; & \breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n} \left(\bar{c}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - c^{(r_2,s_2)}_{(i-1)\Delta T_n}\right)[2] + \left(\bar{c}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n} - c^{(r_1,s_1)}_{(i-1)\Delta T_n}\right)\left(\bar{c}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - c^{(r_2,s_2)}_{(i-1)\Delta T_n}\right), \end{aligned}$$

where it is obvious that

$$\sup_i E\left[\breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n}\left(\bar{c}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - c^{(r_2,s_2)}_{(i-1)\Delta T_n}\right)[2] | \mathcal{F}_{T_{n,i-1}}\right] = o_p\left(a_n\right),$$

and by Lemma 1, $\sup_i \left\|\bar{c}^{(r,s)}_{\Delta T_n,(i-1)\Delta T_n} - c^{(r,s)}_{(i-1)\Delta T_n}\right\|_2 = O_p\left(\Delta T_n^{1/2}\right)$ and $\sup_i \left\|\breve{\beta}^{(r,s)}_{\Delta T_n,(i-1)\Delta T_n}\right\|_2 = O_p\left(a_n \Delta T_n^{-1/2}\right)$, and thus, we obtain:

$$R^{\text{Adjusted-Bias-II}} = O_p\left(a_n \Delta T_n^{1/2}\right) + O_p\left(\Delta T_n\right) = o_p\left(a_n\right). \tag{E.7}$$

For $R^{\text{Adjusted-Bias-III}}$ and $R^{\text{Adjusted-Bias-IV}}$, we first decompose $\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}$ as follows:

$$\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{i-1}} = \left(\breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n} \breve{\beta}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n} - \bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}}\right) + \left(\bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}} - \hat{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}}\right),$$

where $\bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}}$ is defined in (E.2), and it is straightforward to obtain:

$$\begin{aligned}
&\breve{\beta}^{(r_1,s_1)}_{\Delta T_n,(i-1)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n,(i-1)\Delta T_n}-\bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}}\\
=\;&\frac{1}{\Delta T_n^2}\left(M^{(r_1,s_1)}_{i\Delta T_n}-M^{(r_1,s_1)}_{(i-1/2)\Delta T_n}\right)\left(M^{(r_2,s_2)}_{(i-1/2)\Delta T_n}-M^{(r_2,s_2)}_{(i-1)\Delta T_n}\right)[2]\\
&+\frac{2}{\Delta T_n^2}\left(M^{(r_1,s_1)}_{i\Delta T_n}-M^{(r_1,s_1)}_{(i-1/2)\Delta T_n}\right)\left(\tilde{e}^{(r_2,s_2)}_{(i-1/2)\Delta T_n}-\tilde{e}^{(r_2,s_2)}_{(i-1)\Delta T_n}\right)[2]\\
&+\frac{1}{\Delta T_n^2}\left(\tilde{e}^{(r_1,s_1)}_{i\Delta T_n}-\tilde{e}^{(r_1,s_1)}_{(i-1/2)\Delta T_n}\right)\left(\tilde{e}^{(r_2,s_2)}_{(i-1/2)\Delta T_n}-\tilde{e}^{(r_2,s_2)}_{(i-1)\Delta T_n}\right)[2],
\end{aligned}$$

and

$$\begin{aligned}
\hat{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}}-\bar{\varphi}^{(r_1,r_2,s_1,s_2)}_{\Delta T_n,T_{n,i-1}} &= \frac{1}{4}\psi_i^{(r_1,s_1)}\psi_i^{(r_2,s_2)}-\frac{1}{4}\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,(i-1/2)\Delta T_n}\breve{\beta}^{(r_2,s_2)}_{\Delta T_n/2,(i-1)\Delta T_n}[2]\\
&\quad+\frac{1}{4}\left(\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,(i-1/2)\Delta T_n}-\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,(i-1)\Delta T_n}\right)\psi_i^{(r_2,s_2)}[2],
\end{aligned}$$

where $\psi_i^{(r_1,s_1)}$ is defined in (E.1).

Because we can further simplify $\psi_i^{(r,s)}$ as follows:

$$\psi_i^{(r,s)}=\int_{(i-1/2)\Delta T_n}^{i\Delta T_n}\left(\frac{T_{n,i}-u}{\Delta T_n/2}\right)dc_u^{(r,s)}+\int_{(i-1)\Delta T_n}^{(i-1/2)\Delta T_n}\left(\frac{u-T_{n,i-1}}{\Delta T_n/2}\right)dc_u^{(r,s)},$$

then we know that $\sup_i\left\|\psi_i^{(r,s)}\right\|_2=O_p\left(\Delta T_n^{1/2}\right)$. By Lemma 1, we know that $\sup_{1\le i\le 2B}\left\|\breve{\beta}^{(r_1,s_1)}_{\Delta T_n/2,i\Delta T_n/2}\right\|_2=O_p\left(a_n\Delta T_n^{-1/2}\right)$, $\sup_{1\le i\le 2B}\left\|\Delta T_n^{-1}\left(M^{(r_1,s_1)}_{i\Delta T_n/2}-M^{(r_1,s_1)}_{(i-1)\Delta T_n/2}\right)\right\|_2=O_p\left(a_n\Delta T_n^{-1/2}\right)$ and $\sup_{1\le i\le 2B}\left\|\Delta T_n^{-1}\left(\tilde{e}^{(r_2,s_2)}_{i\Delta T_n/2}-\tilde{e}^{(r_2,s_2)}_{(i-1)\Delta T_n/2}\right)\right\|_2=O_p\left(a_n^2\Delta T_n^{-1}\right)$, which implies that

$$\sup_i\left\|\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}-E\left(\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}|\mathcal{F}_{T_{n,i-1}}\right)\right\|_2=O_p\left(a_n^2\Delta T_n^{-1}\right),$$

and because $a_n^{-1}\Delta T_n\to 0$ and $a_n^{-3/2}\Delta T_n\to\infty$ as $n\to\infty$, we have:

$$\sup_i\left|E\left(\breve{\phi}^{(r_1,r_2,s_1,s_2)}_{T_{n,i-1}}|\mathcal{F}_{T_{n,i-1}}\right)\right|=O_p\left(\Delta T_n\right)+O_p\left(a_n^4\Delta T_n^{-2}\right)=o_p\left(a_n\right).$$

Finally, we obtain:

$$R^{\text{Adjusted-Bias-III}} = O_p\left(a_n^2\Delta T_n^{-1/2}\right)=o_p\left(a_n\right), \tag{E.8}$$

$$R^{\text{Adjusted-Bias-IV}} = o_p\left(a_n\right). \tag{E.9}$$

Plugging (E.6)-(E.9) into (E.4), we obtain:

$$R^{\text{Adjusted-Bias}} = o_p\left(a_n\right). \tag{E.10}$$

Plugging (E.5) and (E.10) into (E.3), we finally obtain:

$$\tilde{V}\left(\Delta T_n, X; F_m\right) - \int_0^{\mathcal{T}} F_m\left(c_s\right) ds = R^{\text{Spot-V}} + o_p\left(a_n\right) = O_p\left(a_n\right).$$

Recall the definition of $R^{\text{Spot-V}}$ in (4.3), and $\beta^{(r,s)}_{\Delta T_n, T_{n,i-1}}$ and $\tilde{\beta}^{(r,s)}_{\Delta T_n, T_{n,i-1}}$ in (3.5) and (3.3), and by Lemma 1, we have the following decomposition:

$$\beta^{(r,s)}_{\Delta T_n, T_{n,i-1}} = \underbrace{\bar{c}^{(r,s)}_{\Delta T_n, T_{n,i-1}} - c^{(r,s)}_{T_{n,i-1}}}_{O_p\left(\Delta T_n^{1/2}\right)} + \underbrace{\frac{1}{\Delta T_n}\left(M^{(r,s)}_{T_{n,i}} - M^{(r,s)}_{T_{n,i-1}}\right)}_{\tilde{\beta}^{(r,s)}_{\Delta T_n, T_{n,i-1}}} + \underbrace{\frac{1}{\Delta T_n}\left(\tilde{e}^{(r,s)}_{T_{n,i}} - \tilde{e}^{(r,s)}_{T_{n,i-1}}\right)}_{O_p\left(a_n^2 \Delta T_n^{-1}\right)}. \tag{E.11}$$

Therefore, we obtain:

$$R^{\text{Spot-V}} - \Delta T_n \sum_{i=1}^{B}\left[\sum_{r_1,s_1=1}^{d} \partial_{r_1 s_1} F_m\left(c_{T_{n,i-1}}\right) \tilde{\beta}^{(r,s)}_{\Delta T_n, T_{n,i-1}}\right] = O_p\left(\Delta T_n\right) + O_p\left(a_n^2 \Delta T_n^{-1/2}\right) = o_p\left(a_n\right),$$

and finally, the estimation error of the bias corrected estimator could be expressed as:

$$\tilde{V}\left(\Delta T_n, X; F_m\right) - \int_0^{\mathcal{T}} F_m\left(c_s\right) ds = \tilde{R}^{\text{Spot-V}} + o_p\left(a_n\right).$$

with

$$\tilde{R}^{\text{Spot-V}} = \sum_{i=1}^{B}\left[\sum_{r_1,s_1=1}^{d} \partial_{r_1 s_1} F_m\left(c_{T_{n,i-1}}\right)\left(M^{(r,s)}_{T_{n,i}} - M^{(r,s)}_{T_{n,i-1}}\right)\right].$$

If we define $\left[M^{(r_1,s_1)}, M^{(r_2,s_2)}\right]^{(B)}_t$ as (4.5), then we know that the (p,q)-th element of the covariance matrix $\tilde{\Sigma}_n$ of $\tilde{V}\left(\Delta T_n, X; F\right) - \int_0^{\mathcal{T}} F\left(c_s\right) ds$ can be expressed as follows:

$$\tilde{\Sigma}^{(p,q)}_n = \sum_{r_1,s_1,r_2,s_2=1}^{d} \int_0^{\mathcal{T}} \partial_{r_1 s_1} F_p\left(c_u\right) \partial_{r_2 s_2} F_q\left(c_u\right) d\left[M^{(r_1,s_1)}, M^{(r_2,s_2)}\right]^{(B)}_u.$$

Note that $a_n^{-2}\tilde{\Sigma}^{(p,q)}_n \xrightarrow{p} \Sigma^{(p,q)}$, the theorem got proved.

F Proof of Theorems 3, 4 and 5

Before the proof of the main theorems, we first show some preliminary lemmas. As in Assumption 5, we denote the columns of $\mathbf{B}_t^\intercal$ as $\mathbf{b}_t^{(1)}, \mathbf{b}_t^{(2)}, \ldots, \mathbf{b}_t^{(d)}$. We also denote the columns of $\mathbf{B}_t$ as $\tilde{\mathbf{b}}_t^{(1)}, \tilde{\mathbf{b}}_t^{(2)}, \ldots, \tilde{\mathbf{b}}_t^{(q)}$.

LEMMA 4. *If we define* $\hat{c}_t = \left\{\hat{c}_{\Delta T_n,t}^{(r,s)}\right\}_{1\le r,s\le d}$ *with* $\Delta T_n \asymp a_n$ *and* a_n *is defined in (2.6). For basic settings about the observations, we assume Conditions 1-4 in Mykland et al. (2019), and Assumptions 1-3. Then the elementwise max norm of the estimation error has the rate* $\|\hat{c}_t - c_t\|_{\max} = O_p\left((\Delta T_n \log d)^{\frac{1}{2}}\right)$.

Proof. Based on the results of Lemma 1 and 2, we can conclude that there exists positive constants C_1 and C_2, such that for all $1 \le r, s \le d$, and any $x > 0$,

$$P\left(\left|\hat{c}_t^{(r,s)} - c_t^{(r,s)}\right| > x\right) \le C_1 \exp\left(-\frac{C_2 x^2}{\Delta T_n}\right). \tag{F.1}$$

The detailed proof follows from the similar discussion in the proof of Lemma A.1 in Fan et al. (2016a). Because of the fact that

$$\{\|\hat{c}_t - c_t\|_{\max} > x\} = \bigcup_{r,s}\left\{\left|\hat{c}_t^{(r,s)} - c_t^{(r,s)}\right| > x\right\},$$

it follows from the Bonferroni inequality that we can easily obtain the convergence rate, using the similar technique in Lemma A.2 (iv) of Fan et al. (2016a). □

Next, we show the $q-$th largest eigenvalue of the spot covariance matrix estimator diverges with respect to d, where q is the number of common factors.

LEMMA 5. *Denote the* $q-$*th largest eigenvalue of* $\hat{c}_t$ *by* $\hat{\lambda}_t^{(q)}$. *Assume* $\log d = o\left(\Delta T_n^{-1}\right)$, *where* ΔT_n *follows the definition in Lemma 4. Then* $\hat{\lambda}_t^{(q)} > C_3 d$ *with probability approaching* 1 *for some constant* $C_3 > 0$.

Proof. First of all, by Proposition 2 and its assumptions, it is easy to see that the $q-$th largest eigenvalue of c_t, denoted by $\lambda_t^{(q)}$, satisfies that, for some $C_3' > 0$,

$$\lambda_t^{(q)} \ge \left\|\tilde{\mathbf{b}}_t^{(q)}\right\|^2 - \left|\lambda_t^{(q)} - \left\|\tilde{\mathbf{b}}_t^{(q)}\right\|^2\right| \ge C_3' d - \|\mathbf{s}_t\| \ge \left(\frac{C_3'}{2}\right) d,$$

when d is large enough. This is because $\|\mathbf{s}_t\|$ is bounded with respect to d. Next, by Weyl's theorem, we just need to show that $\|\hat{c}_t - c_t\| = o_p(d)$. Because of the fact that $\|\mathbf{A}\| \le d\,\|\mathbf{A}\|_{\max}$ for $d \times d$ matrix $\mathbf{A}$, and based on the result of Lemma 4, we obtain:

$$\|\hat{c}_t - c_t\| \le d\,\|\hat{c}_t - c_t\|_{\max} = O_p\left(d\,(\Delta T_n \log d)^{\frac{1}{2}}\right) = o_p(d),$$

which follows from the assumption $\log d = o\left(\Delta T_n^{-1}\right)$. This proves the lemma. □

Next, we complete the proof of Theorem 3.

Proof of Theorem 3. Define

$$\mathcal{K}_{n,d} = \left(\Delta T_n \log d\right)^{1/2} + d^{-1},$$

and

$$\begin{aligned} \mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) &= d^{-1}\mathrm{tr}\left(\hat{c}_t - \hat{\mathbf{B}}_{k,t}\hat{\mathbf{B}}_{k,t}^{\intercal}\right), \\ \mathcal{PC}\left(k, \hat{\mathbf{B}}_{k,t}\right) &= \mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) + k\mathcal{G}\left(\Delta T_n, d\right), \end{aligned}$$

where $\hat{\mathbf{B}}_{k,t}$ is as in Definition (5.13). Similarly, we define $\hat{\mathbf{\Lambda}}_{k,t}$ =Diag$\left(\hat{\lambda}_t^{(1)}, \hat{\lambda}_t^{(2)}, \ldots, \hat{\lambda}_t^{(k)}\right)$ and $\hat{\mathbf{\Gamma}}_{k,t} = \left(\hat{\gamma}_t^{(1)}, \hat{\gamma}_t^{(2)}, \ldots, \hat{\gamma}_t^{(k)}\right)$, where $\hat{\lambda}_t^{(i)}$ is the $i-$th largest eigenvalue of $\hat{c}_t$, and $\hat{\gamma}_t^{(i)}$ is the corresponding eigenvector.

Observe that:

$$\mathcal{PC}\left(k, \hat{\mathbf{B}}_{k,t}\right) - \mathcal{PC}\left(q, \hat{\mathbf{B}}_{q,t}\right) = \mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) - \mathcal{V}\left(q, \hat{\mathbf{B}}_{q,t}\right) + (k-q)\,\mathcal{G}\left(\Delta T_n, d\right), \tag{F.2}$$

where

$$\mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) - \mathcal{V}\left(q, \hat{\mathbf{B}}_{q,t}\right) = d^{-1}\mathrm{tr}\left(\hat{\mathbf{B}}_{q,t}\hat{\mathbf{B}}_{q,t}^{\intercal} - \hat{\mathbf{B}}_{k,t}\hat{\mathbf{B}}_{k,t}^{\intercal}\right).$$

We first show that $P\left(\mathcal{PC}\left(k, \hat{\mathbf{B}}_{k,t}\right) < \mathcal{PC}\left(q, \hat{\mathbf{B}}_{q,t}\right)\right) \to 0$ for $k < q$. Because $\mathrm{tr}\left(\hat{\mathbf{B}}_{q,t}\hat{\mathbf{B}}_{q,t}^{\intercal}\right)$ =$\mathrm{tr}\left(\hat{\mathbf{B}}_{q,t}^{\intercal}\hat{\mathbf{B}}_{q,t}\right)$, we have for $C_3 > 0$,

$$\mathrm{tr}\left(\hat{\mathbf{B}}_{q,t}\hat{\mathbf{B}}_{q,t}^{\intercal} - \hat{\mathbf{B}}_{k,t}\hat{\mathbf{B}}_{k,t}^{\intercal}\right) = \mathrm{tr}\left(\hat{\mathbf{\Lambda}}_{q,t}\right) - \mathrm{tr}\left(\hat{\mathbf{\Lambda}}_{k,t}\right) = \sum_{i=k+1}^{q} \hat{\lambda}_t^{(i)} \geq \hat{\lambda}_t^{(q)} > C_3 d,$$

with probability approaching 1, which follows from the result of Lemma 5. It is then easy to see that

$$\mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) - \mathcal{V}\left(q, \hat{\mathbf{B}}_{q,t}\right) > C_3 > 0, \tag{F.3}$$

with probability approaching 1. Moreover, because $k \leq q_{\max}$ and $(k-q)\,\mathcal{G}\left(\Delta T_n, d\right) \to 0$, the statement is proved for $k < q$.

Second, we show $P\left(\mathcal{PC}\left(k, \hat{\mathbf{B}}_{k,t}\right) < \mathcal{PC}\left(q, \hat{\mathbf{B}}_{q,t}\right)\right) \to 0$ for $k > q$. Because $\mathcal{V}\left(k, \hat{\mathbf{B}}_{k,t}\right) - \mathcal{V}\left(q, \hat{\mathbf{B}}_{q,t}\right) =$

$-d^{-1}\sum_{i=q+1}^{k}\hat{\lambda}_t^{(i)}$, we have:

$$\left|\mathcal{V}\left(k,\hat{\mathbf{B}}_{k,t}\right)-\mathcal{V}\left(q,\hat{\mathbf{B}}_{q,t}\right)\right|\leq d^{-1}\sum_{i=q+1}^{k}\left|\hat{\lambda}_t^{(i)}-\lambda_t^{(i)}\right|+d^{-1}\sum_{i=q+1}^{k}\lambda_t^{(i)},$$

where the first term on the right hand side can be bounded by Weyl's theorem and the fact that $\|\mathbf{A}\|\leq d\,\|\mathbf{A}\|_{\max}$ for a $d\times d$ matrix $\mathbf{A}$:

$$d^{-1}\sum_{i=q+1}^{k}\left|\hat{\lambda}_t^{(i)}-\lambda_t^{(i)}\right|\leq d^{-1}\left(k-q\right)\|\hat{c}_t-c_t\|\leq 2q_{\max}\|\hat{c}_t-c_t\|_{\max},$$

while the second term can be bounded similarly using Weyl's theorem:

$$d^{-1}\sum_{i=q+1}^{k}\lambda_t^{(i)}\leq d^{-1}\left(k-q\right)\lambda_t^{(q+1)}\leq d^{-1}q_{\max}\|\mathbf{s}_t\|.$$

Based on the result of Lemma 4, and Assumption 5, we know that $\|\mathbf{s}_t\|\leq\|\mathbf{s}_t\|_1<\vartheta_2$, and consequently

$$\mathcal{V}\left(q,\hat{\mathbf{B}}_{q,t}\right)-\mathcal{V}\left(k,\hat{\mathbf{B}}_{k,t}\right)=O_p\left(\mathcal{K}_{n,d}\right).$$

for $q<k<q_{\max}$. From the assumption that $\mathcal{K}_{n,d}^{-1}\mathcal{G}\left(\Delta T_n,d\right)\to\infty$, and noting that

$$P\left(\mathcal{PC}\left(k,\hat{\mathbf{B}}_{k,t}\right)<\mathcal{PC}\left(q,\hat{\mathbf{B}}_{q,t}\right)\right)=P\left(\mathcal{V}\left(q,\hat{\mathbf{B}}_{q,t}\right)-\mathcal{V}\left(k,\hat{\mathbf{B}}_{k,t}\right)>\left(k-q\right)\mathcal{G}\left(\Delta T_n,d\right)\right),$$

we can conclude that for $q<k<q_{\max}$, $P\left(\mathcal{PC}\left(k,\hat{\mathbf{B}}_{k,t}\right)<\mathcal{PC}\left(q,\hat{\mathbf{B}}_{q,t}\right)\right)\to 0$. □

F. 1 Results by conditioning on $\hat{q}_t=q$

In view of Theorem 3, all the subsequent results and related proofs will be conditioning on

$$\hat{q}_t=q.$$

Without loss of generality, from now on, we omit the subscript $\hat{q}_t$ is the notation, for example, denote $\hat{\mathbf{B}}_{\hat{q}_t,t},\hat{\mathbf{\Gamma}}_{\hat{q}_t,t},\hat{\mathbf{\Lambda}}_{\hat{q}_t,t},\hat{\mathbf{s}}^*_{\hat{q}_t,t}$ and $\hat{c}^*_{\hat{q}_t,t}$ by $\hat{\mathbf{B}}_t,\hat{\mathbf{\Gamma}}_t,\hat{\mathbf{\Lambda}}_t,\hat{\mathbf{s}}^*_t$ and $\hat{c}^*_t$, respectively.

Following definition (5.10), we denote the columns of $\hat{\mathbf{B}}_t^{\intercal}$ as $\hat{\mathbf{b}}_t^{(1)},\hat{\mathbf{b}}_t^{(2)},\ldots,\hat{\mathbf{b}}_t^{(d)}$. Thus, $\hat{\mathbf{B}}_t^{\intercal}=\left(\hat{\mathbf{b}}_t^{(1)},\hat{\mathbf{b}}_t^{(2)},\ldots,\hat{\mathbf{b}}_t^{(d)}\right)$.

Also recall that $\mathbf{B}_t^\intercal = \left(\mathbf{b}_t^{(1)}, \mathbf{b}_t^{(2)}, \ldots, \mathbf{b}_t^{(d)}\right)$. Define a transition matrix

$$\mathbf{H}_t = \hat{\mathbf{\Lambda}}_t^{-1/2} \hat{\mathbf{\Gamma}}_t^\intercal \mathbf{\Gamma}_t \mathbf{\Lambda}_t^{1/2}, \tag{F.4}$$

and recall the definition of the projection matrix $\mathbf{P_A}$ in formula (5.7). Define

$$\mathbf{V}_t = \hat{\mathbf{\Gamma}}_t \hat{\mathbf{\Gamma}}_t^\intercal, \tag{F.5}$$

and note that $\hat{\mathbf{B}}_t^\intercal = \hat{\mathbf{\Lambda}}_t^{1/2} \hat{\mathbf{\Gamma}}_t^\intercal$. Consequently we have:

$$\mathbf{P}_{\hat{\mathbf{B}}_t} = \mathbb{I}_d - \hat{\mathbf{B}}_t \hat{\mathbf{\Lambda}}_t^{-1} \hat{\mathbf{B}}_t^\intercal = \mathbb{I}_d - \mathbf{V}_t. \tag{F.6}$$

LEMMA 6. *We have the following identities:*

(i)

$$\hat{\mathbf{s}}_t - \mathbf{s}_t = \mathbf{P}_{\hat{\mathbf{B}}_t} (\hat{c}_t - c_t) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal + \mathbf{P}_{\hat{\mathbf{B}}_t} \left(\mathbf{B}_t \mathbf{B}_t^\intercal - \hat{\mathbf{B}}_t \hat{\mathbf{B}}_t^\intercal\right) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal - \mathbf{s}_t \mathbf{V}_t^\intercal - \mathbf{V}_t \mathbf{s}_t + \mathbf{V}_t \mathbf{s}_t \mathbf{V}_t^\intercal, \tag{F.7}$$

(ii)

$$\hat{\mathbf{B}}_t^\intercal - \mathbf{H}_t \mathbf{B}_t^\intercal = \hat{\mathbf{\Lambda}}_t^{-1/2} \hat{\mathbf{\Gamma}}_t^\intercal \left[(\hat{c}_t - c_t) + \mathbf{s}_t \mathbf{V}_t^\intercal + \mathbf{V}_t \mathbf{s}_t - \mathbf{V}_t \mathbf{s}_t \mathbf{V}_t^\intercal\right], \text{and} \tag{F.8}$$

(iii)

$$\mathbf{H}_t \mathbf{H}_t^\intercal - \mathbb{I}_{\hat{q}_t} = \hat{\mathbf{\Lambda}}_t^{-1/2} \hat{\mathbf{\Gamma}}_t^\intercal \left[\mathbf{V}_t \mathbf{s}_t \mathbf{V}_t^\intercal - \mathbf{s}_t \mathbf{V}_t^\intercal - \mathbf{V}_t \mathbf{s}_t - (\hat{c}_t - c_t)\right] \hat{\mathbf{\Gamma}}_t \hat{\mathbf{\Lambda}}_t^{-1/2}. \tag{F.9}$$

Proof. (i) In view of the identities and related derivation of (5.8) and (5.9), we have the following fact:

$$\hat{\mathbf{s}}_t = \mathbf{P}_{\hat{\mathbf{B}}_t} \hat{c}_t \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal. \tag{F.10}$$

This equality can be further decomposed based on (F.6) and $c_t = \mathbf{B}_t \mathbf{B}_t^\intercal + \mathbf{s}_t$, as follows:

$$\hat{\mathbf{s}}_t = \mathbf{P}_{\hat{\mathbf{B}}_t} (\hat{c}_t - c_t) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal + \mathbf{P}_{\hat{\mathbf{B}}_t} (\mathbf{B}_t \mathbf{B}_t^\intercal) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal + \mathbf{P}_{\hat{\mathbf{B}}_t} \mathbf{s}_t \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal. \tag{F.11}$$

In the above equation, the second term on the right hand side can be simplified as:

$$\mathbf{P}_{\hat{\mathbf{B}}_t} (\mathbf{B}_t \mathbf{B}_t^\intercal) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal = \mathbf{P}_{\hat{\mathbf{B}}_t} \left(\mathbf{B}_t \mathbf{B}_t^\intercal - \hat{\mathbf{B}}_t \hat{\mathbf{B}}_t^\intercal\right) \mathbf{P}_{\hat{\mathbf{B}}_t}^\intercal, \tag{F.12}$$

because of the fact that $\mathbf{P}_{\hat{\mathbf{B}}_t}\hat{\mathbf{B}}_t = 0$, while the third term can be further decomposed as:

$$\begin{aligned}\mathbf{P}_{\hat{\mathbf{B}}_t}\mathbf{s}_t\mathbf{P}_{\hat{\mathbf{B}}_t}^{\intercal} &= (\mathbb{I}_d - \mathbf{V}_t)\,\mathbf{s}_t\,(\mathbb{I}_d - \mathbf{V}_t)^{\intercal} \\ &= \mathbf{s}_t - \mathbf{s}_t\mathbf{V}_t^{\intercal} - \mathbf{V}_t\mathbf{s}_t + \mathbf{V}_t\mathbf{s}_t\mathbf{V}_t^{\intercal}, \end{aligned} \tag{F.13}$$

using formula (F.6). Combing (F.10)-(F.13), we obtain (F.7).

(ii) Recalling the definitions $\hat{\mathbf{B}}_t^{\intercal} = \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}$, $\mathbf{B}_t^{\intercal} = \mathbf{\Lambda}_t^{1/2}\mathbf{\Gamma}_t^{\intercal}$, as well as (F.4), we have:

$$\begin{aligned}\hat{\mathbf{B}}_t^{\intercal} - \mathbf{H}_t\mathbf{B}_t^{\intercal} &= \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal} - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\mathbf{\Gamma}_t\mathbf{\Lambda}_t\mathbf{\Gamma}_t^{\intercal}\right) \\ &= \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal} - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\mathbf{B}_t\mathbf{B}_t^{\intercal} \\ &= \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal} - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(c_t - \hat{c}_t + \hat{c}_t - \hat{\mathbf{s}}_t + \hat{\mathbf{s}}_t - \mathbf{s}_t\right) \\ &= \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal} + \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - c_t\right) - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right) - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - \hat{\mathbf{s}}_t\right), \end{aligned} \tag{F.14}$$

where, in view of $\hat{c}_t - \hat{\mathbf{s}}_t = \hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^{\intercal} = \hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t\hat{\mathbf{\Gamma}}_t^{\intercal}$ and $\hat{\mathbf{\Gamma}}_t^{\intercal}\hat{\mathbf{\Gamma}}_t = \mathbb{I}_q$, we have $\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - \hat{\mathbf{s}}_t\right) = \hat{\mathbf{\Lambda}}_t^{1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}$. We then obtain:

$$\hat{\mathbf{B}}_t^{\intercal} - \mathbf{H}_t\mathbf{B}_t^{\intercal} = \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - c_t\right) - \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right). \tag{F.15}$$

On the other hand, observing that $\mathbf{P}_{\hat{\mathbf{B}}_t} = \mathbf{P}_{\hat{\mathbf{\Gamma}}_t}$ and $\mathbf{P}_{\mathbf{B}_t} = \mathbf{P}_{\mathbf{\Gamma}_t}$, then substituting (F.7) into (F.15), we obtain (F.8), based on the fact that $\hat{\mathbf{\Gamma}}_t^{\intercal}\mathbf{P}_{\hat{\mathbf{\Gamma}}_t} = \mathbf{P}_{\hat{\mathbf{\Gamma}}_t}\hat{\mathbf{\Gamma}}_t = 0$.

(iii) Based on a similar derivation as (F.14), and recalling the definition (F.4), we obtain:

$$\mathbf{H}_t\mathbf{H}_t^{\intercal} - \mathbb{I}_{\hat{q}_t} = \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(c_t - \hat{c}_t\right)\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2} + \hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{\mathbf{s}}_t - \mathbf{s}_t\right)\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2}. \tag{F.16}$$

Then substituting (F.7) into (F.16), we obtain (F.9) by using the similar techniques as in (ii). □

Recall the definition $\hat{\mathbf{B}}_t^{\intercal} = \left(\hat{\mathbf{b}}_t^{(1)}, \hat{\mathbf{b}}_t^{(2)}, \ldots, \hat{\mathbf{b}}_t^{(d)}\right)$, whereby $\hat{\mathbf{b}}_t^{(i)} = \left(\hat{\mathbf{B}}_t^{\intercal}\right)_{\bullet,i}$. The $i-$th column of $\hat{\mathbf{B}}_t^{\intercal} - \mathbf{H}_t\mathbf{B}_t^{\intercal}$ can then be expressed as $\left(\hat{\mathbf{B}}_t^{\intercal} - \mathbf{H}_t\mathbf{B}_t^{\intercal}\right)_{\bullet,i} = \hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}$. Further define:

$$\check{\mathbf{s}}_t = \mathbf{V}_t\mathbf{s}_t\mathbf{V}_t^{\intercal} - \mathbf{s}_t\mathbf{V}_t^{\intercal} - \mathbf{V}_t\mathbf{s}_t. \tag{F.17}$$

Also define $\mathbf{e}_j$ to be the row vector for which the $j-$th element equals 1, and the others equal zero. Then for any matrix $\mathbf{A}$, its $i-$th row can be expressed as $(\mathbf{A})_{i,\bullet} = \mathbf{e}_i\mathbf{A}$, while its $j-$th column has the form $(\mathbf{A})_{\bullet,j} = \mathbf{A}\mathbf{e}_j^{\intercal}$. We then have the following preliminary lemma.

LEMMA 7. *We have the following results:*

(i)

$$\|\check{\mathbf{s}}_t\| \leq 3\,\|\mathbf{s}_t\|\,, \tag{F.18}$$

(ii)

$$\max_{1\leq i\leq d}\left\|\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right\| = O_p\left(\|\hat{c}_t - c_t\|_{\max} + d^{-1/2}\,\|\mathbf{s}_t\|\right), \text{and} \tag{F.19}$$

(iii)

$$\|\mathbf{H}_t\mathbf{H}_t^{\intercal} - \mathbb{I}_{\hat{q}_t}\| = O_p\left(\|\hat{c}_t - c_t\|_{\max} + d^{-1}\,\|\mathbf{s}_t\|\right). \tag{F.20}$$

Proof. (i) Recalling the definition (F.17), and by the properties of the spectral norm, we obtain:

$$\|\check{\mathbf{s}}_t\| \leq \|\mathbf{V}_t\|^2\,\|\mathbf{s}_t\| + 2\,\|\mathbf{V}_t\|\,\|\mathbf{s}_t\| = 3\,\|\mathbf{s}_t\|\,,$$

since $\|\mathbf{V}_t\| = \|\mathbf{V}_t^{\intercal}\| = 1$.

(ii) Because $\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}$ is the $i-$th column of $\hat{\mathbf{B}}_t^{\intercal} - \mathbf{H}_t\mathbf{B}_t^{\intercal}$, then by identity (F.8), we have:

$$\max_{1\leq i\leq d}\left\|\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right\| \leq \max_{1\leq i\leq d}\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\,(\hat{c}_t - c_t)\,\mathbf{e}_i^{\intercal}\right\| + \max_{1\leq i\leq d}\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\check{\mathbf{s}}_t\mathbf{e}_i^{\intercal}\right\|. \tag{F.21}$$

The first term on the right hand side of (F.21) can bounded as follows. Since the Cauchy-Schwarz inequality assures $\|\mathbf{A}\mathbf{x}\| \leq \|\mathbf{A}\|_{\mathrm{F}}\,\|\mathbf{x}\|$ for a matrix $\mathbf{A}$ and a vector $\mathbf{x}$, we obtain:

$$\begin{aligned}
\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\,(\hat{c}_t - c_t)\,\mathbf{e}_i^{\intercal}\right\| &\leq \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\right\|_{\mathrm{F}}\,\|(\hat{c}_t - c_t)\,\mathbf{e}_i^{\intercal}\| \\
&= \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\right\|_{\mathrm{F}}\,\|(\hat{c}_t - c_t)\,\mathbf{e}_i^{\intercal}\| \\
&\leq \left\|\left(d^{-1}\hat{\mathbf{\Lambda}}_t\right)^{-1/2}\right\|_{\mathrm{F}}\,\|\hat{c}_t - c_t\|_{\max}
\end{aligned}$$

in view of the facts that $\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\right\|_{\mathrm{F}} = \left(\sum_{l=1}^{q}\left(\hat{\lambda}_t^{(l)}\right)^{-1}\right)^{1/2}$ and $\|\mathbf{A}\| \leq \sqrt{pq}\,\|\mathbf{A}\|_{\max}$ for a matrix $\mathbf{A}$ of dimension $p\times q$. Based on the result of Lemma 5, we know that there exists some $C_3 > 0$ such that $\left\|\left(d^{-1}\hat{\mathbf{\Lambda}}_t\right)^{-1/2}\right\|_{\mathrm{F}} \leq q^{1/2}C_3^{-1/2}$, and consequently, we obtain that:

$$\max_{1\leq i\leq d}\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\,(\hat{c}_t - c_t)\,\mathbf{e}_i^{\intercal}\right\| \leq q^{1/2}C_3^{-1/2}\,\|\hat{c}_t - c_t\|_{\max}\,,$$

with probability approaching 1.

For the second term on the right hand side of (F.21), we have:

$$\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\check{\mathbf{s}}_t\mathbf{e}_i^{\intercal}\right\| \leq d^{-1/2}\left\|\left(d^{-1}\hat{\mathbf{\Lambda}}_t\right)^{-1/2}\right\|\left\|\hat{\mathbf{\Gamma}}_t^{\intercal}\right\|\|\check{\mathbf{s}}_t\|\,\|\mathbf{e}_i^{\intercal}\|\,.$$

Since $\|\mathbf{e}_i^{\intercal}\| = \left\|\hat{\mathbf{\Gamma}}_t^{\intercal}\right\| = 1$, and by Lemma 5, we have $\left\|\left(d^{-1}\hat{\mathbf{\Lambda}}_t\right)^{-1/2}\right\| \leq C_3^{-1/2}$ with probability approaching 1. Also recall the result in (i) to obtain:

$$\max_{1\leq i\leq d}\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\check{\mathbf{s}}_t\mathbf{e}_i^{\intercal}\right\| \leq 3C_3^{-1/2}d^{-1/2}\|\mathbf{s}_t\|\,.$$

Therefore, we obtain (F.19).

(iii) Conditioning on $\hat{q}_t = q$. Recall the identity (F.9), by triangle inequality, we obtain:

$$\|\mathbf{H}_t\mathbf{H}_t^{\intercal} - \mathbb{I}_q\| \leq \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\check{\mathbf{s}}_t\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2}\right\| + \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - c_t\right)\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2}\right\|,$$

where the first term on the right hand side can be bounded as follows:

$$\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\check{\mathbf{s}}_t\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2}\right\| \leq \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\right\|^2\|\check{\mathbf{s}}_t\| \leq 3C_3^{-1}d^{-1}\|\mathbf{s}_t\|\,,$$

with probability approaching 1, while the second term on the right hand side has the following bound:

$$\left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\hat{\mathbf{\Gamma}}_t^{\intercal}\left(\hat{c}_t - c_t\right)\hat{\mathbf{\Gamma}}_t\hat{\mathbf{\Lambda}}_t^{-1/2}\right\| \leq \left\|\hat{\mathbf{\Lambda}}_t^{-1/2}\right\|^2\|\hat{c}_t - c_t\| \leq C_3^{-1}d^{-1}\|\hat{c}_t - c_t\| \leq C_3^{-1}\|\hat{c}_t - c_t\|_{\max}\,,$$

where the last inequalities is based on the fact that $\|\mathbf{A}\| \leq d\|\mathbf{A}\|_{\max}$ for a $d\times d$ matrix $\mathbf{A}$. Finally the result (F.20) is proved. □

Proof of Theorem 4. Recall that $\hat{\mathbf{B}}_t^{\intercal} = \left(\hat{\mathbf{b}}_t^{(1)}, \hat{\mathbf{b}}_t^{(2)}, \ldots, \hat{\mathbf{b}}_t^{(d)}\right)$, and hence the (i,j) −th element of $\hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^{\intercal}$ can be expressed as $\left(\hat{\mathbf{b}}_t^{(i)}\right)^{\intercal}\hat{\mathbf{b}}_t^{(j)}$. Consequently, the (i,j) −th element of $\hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^{\intercal} - \mathbf{B}_t\mathbf{B}_t^{\intercal}$ is $\left(\hat{\mathbf{b}}_t^{(i)}\right)^{\intercal}\hat{\mathbf{b}}_t^{(j)} - \left(\mathbf{b}_t^{(i)}\right)^{\intercal}\mathbf{b}_t^{(j)}$. By definition (F.4), we obtain the following identity:

$$\begin{aligned}\left(\hat{\mathbf{b}}_t^{(i)}\right)^{\intercal}\hat{\mathbf{b}}_t^{(j)} - \left(\mathbf{b}_t^{(i)}\right)^{\intercal}\mathbf{b}_t^{(j)} \quad = \quad & \left(\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right)^{\intercal}\left(\hat{\mathbf{b}}_t^{(j)} - \mathbf{H}_t\mathbf{b}_t^{(j)}\right) + \left(\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right)^{\intercal}\mathbf{H}_t\mathbf{b}_t^{(j)} \\ & + \left(\mathbf{H}_t\mathbf{b}_t^{(i)}\right)^{\intercal}\left(\hat{\mathbf{b}}_t^{(j)} - \mathbf{H}_t\mathbf{b}_t^{(j)}\right) + \left(\mathbf{b}_t^{(i)}\right)^{\intercal}\left(\mathbf{H}_t^{\intercal}\mathbf{H}_t - \mathbb{I}_q\right)\mathbf{b}_t^{(j)}.\end{aligned}$$

By triangular inequality, we have:

$$\begin{aligned}
&\left\|\hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^\intercal - \mathbf{B}_t\mathbf{B}_t^\intercal\right\|_{\max} \\
= &\max_{1\le i,j\le d}\left|\left(\hat{\mathbf{b}}_t^{(i)}\right)^\intercal\hat{\mathbf{b}}_t^{(j)} - \left(\mathbf{b}_t^{(i)}\right)^\intercal\mathbf{b}_t^{(j)}\right| \\
\le &\left(\max_{1\le i\le d}\left\|\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right\|\right)^2 + 2\max_{1\le i,j\le d}\left\|\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}\right\|\left\|\mathbf{H}_t\mathbf{b}_t^{(j)}\right\| + \left(\max_{1\le i\le d}\left\|\mathbf{b}_t^{(i)}\right\|\right)^2\left\|\mathbf{H}_t^\intercal\mathbf{H}_t - \mathbb{I}_q\right\|.
\end{aligned}$$

Then based on the Assumptions 4 and 5, we know that

$$\max_{1\le i\le d}\left\|\mathbf{H}_t\mathbf{b}_t^{(i)}\right\| = O_p(1) \text{ and } \max_{1\le i\le d}\left\|\mathbf{b}_t^{(i)}\right\| = O_p(1).$$

On the other hand, based on the result (iii) in Lemma 7, and following the similar discussion of the proof for Lemma 11 (b) in Fan et al. (2013), by conditioning on $\hat{q}_t = q$, we obtain:

$$\mathbf{H}_t^\intercal\mathbf{H}_t - \mathbb{I}_q = O_p\left(\left\|\hat{c}_t - c_t\right\|_{\max} + d^{-1}\left\|\mathbf{s}_t\right\|\right).$$

Finally, recall result (ii) in Lemma 7 to obtain:

$$\left\|\hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^\intercal - \mathbf{B}_t\mathbf{B}_t^\intercal\right\|_{\max} = O_p\left(\left\|\hat{c}_t - c_t\right\|_{\max} + d^{-1/2}\left\|\mathbf{s}_t\right\|\right).$$

On the other hand, because of the identity $\hat{c}_t - c_t = \hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^\intercal - \mathbf{B}_t\mathbf{B}_t^\intercal + \hat{\mathbf{s}}_t - \mathbf{s}_t$, we obtain:

$$\begin{aligned}
\left\|\hat{\mathbf{s}}_t - \mathbf{s}_t\right\|_{\max} &\le \left\|\hat{c}_t - c_t\right\|_{\max} + \left\|\hat{\mathbf{B}}_t\hat{\mathbf{B}}_t^\intercal - \mathbf{B}_t\mathbf{B}_t^\intercal\right\|_{\max} \\
&= O_p\left(\left\|\hat{c}_t - c_t\right\|_{\max} + d^{-1/2}\left\|\mathbf{s}_t\right\|\right).
\end{aligned}$$

Based on the result of Lemma 4 and noting that $\|\mathbf{s}_t\| \le \|\mathbf{s}_t\|_1 < \vartheta_2$ by Assumption 5, the theorem is proved. □

Before the proof of the convergence rateof the precision matrix estimator, we first introduce some preliminary results, which are parallel to Lemmae 14 and 15 in Fan et al. (2013). Define

$$\mathbf{\Phi}_t = \hat{\mathbf{B}}_t^\intercal - \mathbf{H}_t\mathbf{B}_t^\intercal. \tag{F.22}$$

LEMMA 8. *Assume that $\omega_n^{1-\nu}m_d = o(1)$, then with probability approaching 1, there exists some $C_4 > 0$ such that*

(i) $\|\mathbf{\Phi}_t\|_F^2 = O_p\left(d\omega_n^2\right)$,

(ii) $\left\|\hat{\mathbf{B}}_t^\intercal\left(\hat{\mathbf{s}}_t^*\right)^{-1}\hat{\mathbf{B}}_t - \mathbf{H}_t\mathbf{B}_t^\intercal\left(\mathbf{s}_t^{-1}\right)\mathbf{B}_t\mathbf{H}_t^\intercal\right\| = O_p\left(d\omega_n^{1-\nu}m_d\right)$,

(iii) $\lambda_{\min}\left(\mathbb{I}_q + \mathbf{H}_t\mathbf{B}_t^\intercal\left(\mathbf{s}_t^{-1}\right)\mathbf{B}_t\mathbf{H}_t^\intercal\right) \geq C_4 d$,

(iv) $\lambda_{\min}\left(\mathbb{I}_q + \hat{\mathbf{B}}_t^\intercal\left(\hat{\mathbf{s}}_t^*\right)^{-1}\hat{\mathbf{B}}_t\right) \geq C_4 d$,

(v) $\lambda_{\min}\left(\mathbb{I}_q + \mathbf{B}_t^\intercal\mathbf{s}_t^{-1}\mathbf{B}_t\right) \geq C_4 d$, *and*

(vi) $\lambda_{\min}\left(\left(\mathbf{H}_t\mathbf{H}_t^\intercal\right)^{-1} + \mathbf{B}_t^\intercal\mathbf{s}_t^{-1}\mathbf{B}_t\right) \geq C_4 d$.

Proof. We condition on $\hat{q}_t = q$. Recall that $\hat{\mathbf{b}}_t^{(i)} - \mathbf{H}_t\mathbf{b}_t^{(i)}$ is the $i-$th column of $\hat{\mathbf{B}}_t^\intercal - \mathbf{H}_t\mathbf{B}_t^\intercal$. Then, by the result (ii) of Lemma 7, it is easy to verify (i). Result (i) implies result (ii) by following the similar proof of Lemma 14 in Fan et al. (2013). By the result (iii) of Lemma 7, following the similar proof in Lemma 15(a) of Fan et al. (2013), we obtain (iii). The result (iv) follows from (ii) and (iii). The results (v) and (vi) follows from a similar argument as Lemma 15(a) of Fan et al. (2013) and based on result (iii) of Lemma 7. □

Proof of Theorem 5. Define $\tilde{c}_t^* = \mathbf{B}_t\mathbf{H}_t^\intercal\mathbf{H}_t\mathbf{B}_t^\intercal + \mathbf{s}_t$, and also define

$$\begin{aligned}
\mathbf{G}_t &= \left(\mathbb{I}_q + \hat{\mathbf{B}}_t^\intercal\left(\hat{\mathbf{s}}_t^*\right)^{-1}\hat{\mathbf{B}}_t\right)^{-1}, \\
\tilde{\mathbf{G}}_t &= \left(\mathbb{I}_q + \mathbf{H}_t\mathbf{B}_t^\intercal\left(\mathbf{s}_t^{-1}\right)\mathbf{B}_t\mathbf{H}_t^\intercal\right)^{-1},
\end{aligned}$$

then we know that $\left\|\left(\hat{c}_t^*\right)^{-1} - \left(\tilde{c}_t^*\right)^{-1}\right\| \leq \sum_{i=1}^6 L_i$, where

$$\begin{aligned}
L_1 &= \left\|\left(\hat{\mathbf{s}}_t^*\right)^{-1} - \mathbf{s}_t^{-1}\right\|, \\
L_2 &= \left\|\left[\left(\hat{\mathbf{s}}_t^*\right)^{-1} - \mathbf{s}_t^{-1}\right]\hat{\mathbf{B}}_t\mathbf{G}_t\hat{\mathbf{B}}_t^\intercal\left[\left(\hat{\mathbf{s}}_t^*\right)^{-1} - \mathbf{s}_t^{-1}\right]\right\|, \\
L_3 &= 2\left\|\left[\left(\hat{\mathbf{s}}_t^*\right)^{-1} - \mathbf{s}_t^{-1}\right]\hat{\mathbf{B}}_t\mathbf{G}_t\hat{\mathbf{B}}_t^\intercal\mathbf{s}_t^{-1}\right\|, \\
L_4 &= \left\|\mathbf{s}_t^{-1}\mathbf{B}_t\mathbf{H}_t^\intercal\left(\tilde{\mathbf{G}}_t - \mathbf{G}_t\right)\mathbf{H}_t\mathbf{B}_t^\intercal\mathbf{s}_t^{-1}\right\|, \\
L_5 &= \left\|\mathbf{s}_t^{-1}\mathbf{\Phi}_t^\intercal\mathbf{G}_t\mathbf{\Phi}_t\mathbf{s}_t^{-1}\right\|, \text{and} \\
L_6 &= 2\left\|\mathbf{s}_t^{-1}\mathbf{\Phi}_t^\intercal\mathbf{G}_t\mathbf{H}_t\mathbf{B}_t^\intercal\mathbf{s}_t^{-1}\right\|.
\end{aligned}$$

First of all, L_1 is bounded by the result of Proposition 3. By result (iv) of Lemma 8, we have: $\|\mathbf{G}_t\| = O_p\left(d^{-1}\right)$, which implies that $L_3 = O_p\left(L_1\right)$ and $L_2 = o_p\left(L_1\right)$. By the result (i) of Lemma 8, we know that $L_6 = O_p\left(\omega_n\right)$ and $L_5 = o_p\left(\omega_n\right)$. Following from result (iii) of Lemma 8, we have: $\left\|\tilde{\mathbf{G}}_t\right\| = O_p\left(d^{-1}\right)$. Then note that

$$\left\|\tilde{\mathbf{G}}_t - \mathbf{G}_t\right\| = \left\|\tilde{\mathbf{G}}_t\left(\tilde{\mathbf{G}}_t^{-1} - \mathbf{G}_t^{-1}\right)\mathbf{G}_t\right\| \leq O_p\left(d^{-2}\right)\left\|\hat{\mathbf{B}}_t^\intercal\left(\hat{\mathbf{s}}_t^*\right)^{-1}\hat{\mathbf{B}}_t - \mathbf{H}_t\mathbf{B}_t^\intercal\left(\mathbf{s}_t^{-1}\right)\mathbf{B}_t\mathbf{H}_t^\intercal\right\| = O_p\left(d^{-1}\omega_n^{1-\nu}m_d\right),$$

based on result (ii) of Lemma 8. Therefore,

$$\|L_4\| \leq \left\|\mathbf{s}_t^{-1}\mathbf{B}_t\mathbf{H}_t^{\intercal}\right\|^2 \left\|\tilde{\mathbf{G}}_t - \mathbf{G}_t\right\| = O_p\left(\omega_n^{1-\nu} m_d\right).$$

On the other hand, by applying the Sherman-Morrison-Woodbury formula again for $(\tilde{c}_t^*)^{-1}$ and $(c_t)^{-1}$, and based on the results (v) and (vi) in Lemma 8, we obtain:

$$\left\|(\tilde{c}_t^*)^{-1} - (c_t)^{-1}\right\| = o_p\left(\omega_n^{1-\nu} m_d\right),$$

which follows from the similar argument in the proof of subsection C.4.2 of Fan et al. (2013). Finally, by the triangular inequality we obtain:

$$\left\|(\hat{c}_t^*)^{-1} - c_t^{-1}\right\| \leq \left\|(\hat{c}_t^*)^{-1} - (\tilde{c}_t^*)^{-1}\right\| + \left\|(\tilde{c}_t^*)^{-1} - (c_t)^{-1}\right\| = O_p\left(\omega_n^{1-\nu} m_d\right).$$

The theorem is thus proved. □

G More Detailed Simulation Results

G. 1 Simulation Comparison under Different Scenarios In the following, we present more detailed simulation results in the three scenarios described in Section 6.3, where $\Delta\tau_n = 5$, 15, and 60 seconds, respectively.

Table G.1 Simulation Results: Comparison when $\Delta\tau_n = 5$ seconds

		$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ without noise					$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ with noise					$\tilde{V}\left(\Delta T_n, X; F_p^\lambda\right)$		
# Stock	True	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_2
	$p = 1$													
5	0.3852	-0.0003	0.0039	0.003687	0.003815	0.99	2.6566	1.8238	0.029434	0.034323	1.00	-0.0012	0.0193	0.021074
10	0.6729	-0.0007	0.0065	0.006474	0.006771	0.99	5.3605	2.5434	0.058816	0.068704	1.00	0.0025	0.0340	0.036826
20	1.2709	-0.0019	0.0116	0.012252	0.012855	0.99	10.7234	3.6999	0.117462	0.137289	1.00	-0.0015	0.0634	0.070293
30	1.8818	-0.0031	0.0186	0.018141	0.019073	0.99	15.7908	4.3937	0.173089	0.202844	1.00	-0.0052	0.0892	0.101588
50	3.0549	-0.0057	0.0295	0.029685	0.031543	0.99	26.5113	5.7413	0.292705	0.340136	0.99	-0.0032	0.1415	0.167715
	$p = 2$													
5	0.1134	0.0033	0.0032	0.001138	0.001182	0.98	0.2006	0.1093	0.003015	0.002985	0.99	0.0005	0.0079	0.006412
10	0.1735	0.0010	0.0037	0.001741	0.001917	0.93	0.4322	0.1674	0.005845	0.005824	0.99	0.0003	0.0104	0.009882
20	0.2807	-0.0047	0.0075	0.002790	0.003239	0.89	0.9268	0.2483	0.011683	0.011653	0.99	0.0001	0.0155	0.016167
30	0.3909	-0.0093	0.0108	0.003884	0.004548	0.78	1.4306	0.3307	0.017620	0.017623	0.99	0.0004	0.0224	0.022509
50	0.6083	-0.0220	0.0198	0.006080	0.007433	0.90	2.3871	0.4639	0.029237	0.029266	0.99	-0.0006	0.0336	0.035443
	$p = 3$													
5	0.0732	0.0075	0.0045	0.000811	0.000732	0.55	0.0314	0.0209	0.001021	0.000991	0.95	0.0033	0.0085	0.004184
10	0.1028	0.0086	0.0075	0.001196	0.001171	0.51	0.0626	0.0394	0.001639	0.001714	0.97	0.0004	0.0114	0.006005
20	0.1676	-0.0058	0.0066	0.001875	0.001954	0.50	0.1036	0.0633	0.002733	0.003006	0.95	-0.0023	0.0122	0.009736
30	0.2335	-0.0187	0.0087	0.002508	0.002707	0.44	0.1447	0.0823	0.003796	0.004267	0.94	-0.0015	0.0163	0.013438
50	0.3725	-0.0468	0.0200	0.004003	0.004402	0.57	0.2201	0.1148	0.006128	0.007013	0.72	-0.0017	0.0242	0.021385
	$p = 4$													
5	0.0600	-0.0053	0.0021	0.000392	0.000304	0.46	0.0014	0.0041	0.000436	0.000356	0.78	-0.0020	0.0033	0.002033
10	0.0599	-0.0012	0.0009	0.000261	0.000193	0.45	0.0044	0.0024	0.000279	0.000224	0.59	-0.0002	0.0017	0.001203
20	0.0601	0.0007	0.0003	0.000223	0.000152	0.42	0.0055	0.0017	0.000234	0.000176	0.34	0.0000	0.0008	0.000843
30	0.0601	0.0011	0.0005	0.000231	0.000138	0.20	0.0059	0.0015	0.000230	0.000160	0.17	-0.0001	0.0006	0.000711
50	0.0600	0.0016	0.0007	0.000273	0.000132	0.20	0.0061	0.0012	0.000262	0.000153	0.32	-0.0001	0.0005	0.000596

Notes. This table reports the summary statistics for the estimation of the four integrated eigenvalues, i.e., for $p = 1, 2, 3$ and 4, $\int_0^{\mathcal{T}} F_p^\lambda(c_s)ds$ denotes the integrated p-th largest eigenvalue. The Monte Carlo simulation consists of 1000 trials and $\Delta\tau_n = 5$ seconds. The Column "True" denotes the average of true integrated eigenvalue; Column "Bias" denotes the mean of estimation error; Column "Stdev" denotes the standard deviation of the estimation error. "SE_1" denotes the mean of the standard error estimators by plugging $\hat{\lambda}_{T_i}$'s into formula (16) of Corollary 1 in Aït-Sahalia and Xiu (2019). "SE_2" denotes the mean of the standard error estimators constructed as formula (6.2). "Corr" denotes the correlation coefficient between the standard error estimators generated from the columns "SE_1" and "SE_2".

Table G.2 Simulation Results: Comparison when $\Delta\tau_n = 15$ seconds

		$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ without noise					$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ with noise					$\tilde{V}\left(\Delta T_n, X; F_p^\lambda\right)$		
# Stock	True	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_2
	$p = 1$													
5	0.3852	-0.0007	0.0065	0.006496	0.006802	0.98	0.7520	0.5949	0.019426	0.022131	1.00	-0.0010	0.0242	0.025175
10	0.6729	-0.0018	0.0121	0.011519	0.011806	0.99	1.4618	0.8399	0.037315	0.041411	0.99	0.0031	0.0423	0.043862
20	1.2709	-0.0058	0.0206	0.021698	0.022332	0.98	2.8525	1.2350	0.072194	0.081170	0.99	-0.0019	0.0781	0.082760
30	1.8818	-0.0127	0.0323	0.032587	0.035693	0.98	4.1342	1.4673	0.107679	0.124500	0.98	-0.0036	0.1103	0.122126
50	3.0549	-0.0217	0.0533	0.052890	0.057982	0.97	6.9353	1.9426	0.178620	0.207900	0.99	0.0014	0.1725	0.201230
	$p = 2$													
5	0.1134	0.0066	0.0051	0.002112	0.002129	0.80	0.1798	0.1041	0.005038	0.004772	0.96	0.0027	0.0102	0.007795
10	0.1735	0.0062	0.0084	0.003428	0.003456	0.40	0.4112	0.1632	0.010310	0.009165	0.82	0.0021	0.0142	0.012365
20	0.2807	-0.0068	0.0134	0.005301	0.005906	0.65	0.9077	0.2426	0.020833	0.018657	0.92	-0.0015	0.0713	0.020381
30	0.3909	-0.0237	0.0281	0.007970	0.009260	0.35	1.4196	0.3279	0.032759	0.029675	0.69	0.0010	0.0297	0.028373
50	0.6083	-0.0502	0.0438	0.011955	0.014655	0.69	2.3867	0.4606	0.053909	0.049252	0.83	0.0011	0.0463	0.044385
	$p = 3$													
5	0.0732	0.0109	0.0056	0.001520	0.001263	0.51	0.0339	0.0191	0.001862	0.001700	0.89	0.0059	0.0098	0.004916
10	0.1028	0.0246	0.0125	0.002647	0.002094	0.11	0.0718	0.0361	0.003258	0.002977	0.47	0.0046	0.0155	0.007449
20	0.1676	0.0065	0.0145	0.004122	0.003599	0.26	0.1102	0.0614	0.005234	0.005342	0.86	-0.0024	0.0765	0.012203
30	0.2335	-0.0088	0.0255	0.006974	0.005647	0.17	0.1547	0.0832	0.008124	0.008391	0.43	-0.0048	0.0257	0.016794
50	0.3725	-0.0824	0.0273	0.009965	0.008835	0.24	0.2247	0.1203	0.012383	0.013463	0.54	-0.0086	0.0365	0.026863
	$p = 4$													
5	0.0600	-0.0085	0.0025	0.000662	0.000475	0.66	-0.0011	0.0042	0.000753	0.000566	0.65	-0.0047	0.0039	0.002175
10	0.0599	-0.0040	0.0014	0.000492	0.000297	0.28	0.0023	0.0024	0.000547	0.000353	0.22	-0.0012	0.0022	0.001337
20	0.0601	0.0003	0.0005	0.000471	0.000256	0.25	0.0047	0.0015	0.000488	0.000298	0.30	0.0000	0.0020	0.001002
30	0.0601	0.0016	0.0008	0.000549	0.000260	0.21	0.0047	0.0012	0.000564	0.000300	0.19	-0.0001	0.0009	0.000884
50	0.0600	0.0033	0.0011	0.000610	0.000253	0.26	0.0054	0.0010	0.000608	0.000289	0.20	-0.0001	0.0007	0.000784

Notes. This table reports the summary statistics for the estimation of the four integrated eigenvalues, i.e., for $p = 1, 2, 3$ and 4, $\int_0^{\mathcal{T}} F_p^\lambda(c_s)ds$ denotes the integrated p-th largest eigenvalue. The Monte Carlo simulation consists of 1000 trials and $\Delta\tau_n = 15$ seconds. The Column "True" denotes the average of true integrated eigenvalue; Column "Bias" denotes the mean of estimation error; Column "Stdev" denotes the standard deviation of the estimation error. "SE_1" denotes the mean of the standard error estimators by plugging $\hat{\lambda}_{T_i}$'s into formula (16) of Corollary 1 in Aït-Sahalia and Xiu (2019). "SE_2" denotes the mean of the standard error estimators constructed as formula (6.2). "Corr" denotes the correlation coefficient between the standard error estimators generated from the columns "SE_1" and "SE_2".

Table G.3 Simulation Results: Comparison when $\Delta\tau_n = 60$ seconds

		$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ without noise					$\hat{\theta}\left(k_n, \Delta_n, F_p^\lambda\right)$ with noise					$\tilde{V}\left(\Delta T_n, X; F_p^\lambda\right)$		
# Stock	True	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_1	SE_2	Corr	Bias	Stdev	SE_2
	$p = 1$													
5	0.3852	-0.0013	0.0138	0.013781	0.013942	0.98	0.1163	0.1231	0.018269	0.019019	0.93	-0.0064	0.0297	0.028149
10	0.6729	-0.0060	0.0237	0.024039	0.024801	0.97	0.1726	0.1588	0.030962	0.033499	0.93	-0.0063	0.0528	0.049096
20	1.2709	-0.0288	0.0506	0.048097	0.054901	0.81	0.2789	0.2192	0.061616	0.072925	0.73	-0.0232	0.0958	0.092862
30	1.8818	-0.0508	0.0731	0.070762	0.081311	0.82	0.3434	0.2659	0.089415	0.106247	0.42	-0.0265	0.1376	0.135494
50	3.0549	-0.0884	0.1161	0.114829	0.132814	0.63	0.5290	0.3634	0.143709	0.173954	0.58	-0.0468	0.2118	0.222117
	$p = 2$													
5	0.1134	0.0230	0.0102	0.005183	0.004297	0.78	0.1112	0.0633	0.008725	0.006612	0.71	0.0016	0.0143	0.008652
10	0.1735	0.0242	0.0142	0.008241	0.007443	0.58	0.2484	0.1109	0.016760	0.012952	0.76	-0.0016	0.0226	0.013644
20	0.2807	0.0492	0.0398	0.019092	0.015997	0.22	0.5815	0.1655	0.039424	0.028149	0.48	-0.0044	0.0292	0.022589
30	0.3909	0.0140	0.0574	0.026514	0.023128	0.23	0.8892	0.2181	0.059747	0.042254	0.25	-0.0062	0.0412	0.031709
50	0.6083	-0.0653	0.1147	0.041530	0.037052	0.22	1.5007	0.2930	0.098119	0.070033	0.32	-0.0130	0.0674	0.049727
	$p = 3$													
5	0.0732	0.0183	0.0080	0.003574	0.002217	0.38	0.0395	0.0144	0.004345	0.002816	0.56	0.0063	0.0133	0.005426
10	0.1028	0.0536	0.0177	0.006855	0.004349	0.26	0.0955	0.0279	0.008068	0.005694	0.45	0.0043	0.0421	0.008299
20	0.1676	0.1700	0.0482	0.018870	0.009664	0.13	0.2460	0.0494	0.020026	0.012483	0.28	-0.0077	0.0317	0.013656
30	0.2335	0.1978	0.0703	0.028505	0.014294	0.14	0.3159	0.1245	0.031813	0.018627	0.05	-0.0140	0.0400	0.018715
50	0.3725	0.1667	0.1250	0.046640	0.023284	0.07	0.4305	0.1185	0.047476	0.030606	0.27	-0.0205	0.0589	0.029121
	$p = 4$													
5	0.0600	-0.0212	0.0036	0.001171	0.000617	0.30	-0.0148	0.0045	0.001350	0.000733	0.40	-0.0061	0.0048	0.002362
10	0.0599	-0.0106	0.0022	0.001035	0.000515	0.37	-0.0043	0.0026	0.001172	0.000607	0.19	-0.0019	0.0058	0.001488
20	0.0601	-0.0113	0.0019	0.001276	0.000488	0.22	-0.0071	0.0020	0.001383	0.000565	0.16	-0.0006	0.0018	0.001123
30	0.0601	-0.0061	0.0018	0.001478	0.000503	0.08	-0.0034	0.0040	0.001543	0.000574	0.08	-0.0005	0.0013	0.000995
50	0.0600	-0.0004	0.0022	0.001678	0.000518	0.08	-0.0002	0.0014	0.001641	0.000583	0.17	-0.0006	0.0009	0.000884

This table reports the summary statistics for the estimation of the four integrated eigenvalues, i.e., for $p = 1, 2, 3$ and 4, $\int_0^{\mathcal{T}} F_p^\lambda(c_s)ds$ denotes the integrated p-th largest eigenvalue. The Monte Carlo simulation consists of 1000 trials and $\Delta\tau_n = 60$ seconds. The Column "True" denotes the average of true integrated eigenvalue; Column "Bias" denotes the mean of estimation error; Column "Stdev" denotes the standard deviation of the estimation error. "SE_1" denotes the mean of the standard error estimators by plugging $\hat{\lambda}_{T_i}$'s into formula (16) of Corollary 1 in Aït-Sahalia and Xiu (2019). "SE_2" denotes the mean of the standard error estimators constructed as formula (6.2). "Corr" denotes the correlation coefficient between the standard error estimators generated from the columns "SE_1" and "SE_2".

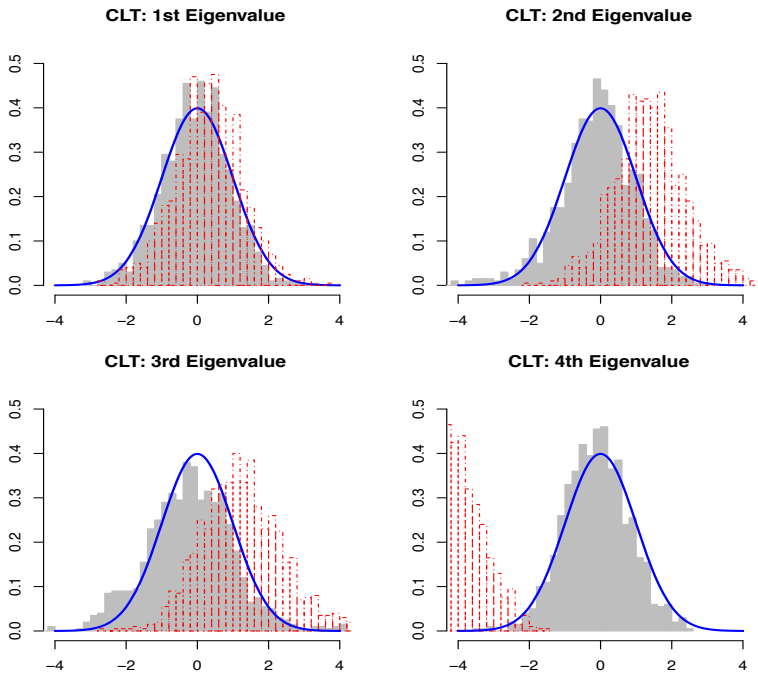

Figure G.1 Finite Sample Distributions of Standardized Statistics

Notes. This figure reports the histogram of the 1000 trials simulation for estimating the four integrated eigenvalues with $\Delta\tau_n = 5$ seconds for 30 stocks over 1 week. The solid blue lines are the standard normal density; the histograms with bars of red dashed border are the distributions of the estimates before bias correction; the gray histograms are the distributions of the estimates after bias correction.

G.2 Distributional performance of the bias-corrected estimator

To validate the asymptotic behavior of the bias corrected estimator, the finite sample distribution of the standardized statistics for $d = 30$ stocks are reported in Figure G.1 where $\Delta\tau_n = 15$ seconds. Note that the standardized statistics are calculated by the following formulas:

$$\frac{\tilde{V}\left(\Delta T_n, X; F_p^\lambda\right) - \int_0^{\mathcal{T}} F_p^\lambda\left(c_s\right) ds}{\widehat{AVAR}\left(\Delta T_n, X; F_p^\lambda\right)^{\frac{1}{2}}},$$

for the standardized statistics of bias-corrected estimator, while

$$\frac{\hat{V}\left(\Delta T_n, X; F_p^\lambda\right) - \int_0^{\mathcal{T}} F_p^\lambda\left(c_s\right) ds}{\widehat{AVAR}\left(\Delta T_n, X; F_p^\lambda\right)^{\frac{1}{2}}},$$

for the standardized statistics of the estimator before bias correction.

Acknowledgments

We would like to thank the Editors, Associate Editor, and the Referees, for comments that substantially improved the paper. Financial support from the National Science Foundation under grants DMS 14-07812 and DMS 17-13129, (Mykland), and DMS 14-07820 and DMS 17-13118, (Zhang) is gratefully acknowledged.

References

Aït-Sahalia, Y. and D. Xiu (2019): "Principal component analysis of high frequency data," *Journal of the American Statistical Association*, 114, 287–303.

Fan, J., A. Furger, and D. Xiu (2016a): "Incorporating Global Industrial Classification Standard Into Portfolio Allocation: A Simple Factor-Based Large Covariance Matrix Estimator With High-Frequency Data," *Journal of Business & Economic Statistics*, 34, 489–503.

Fan, J., Y. Liao, and M. Mincheva (2013): "Large covariance estimation by thresholding principal orthogonal complements," *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 75, 603–680.

Mykland, P. A. and L. Zhang (2006): "ANOVA for diffusions and Ito processes," *The Annals of Statistics*, 34, 1931–1963.

Mykland, P. A., L. Zhang, and D. Chen (2019): "The algebra of two scales estimation, and the S-TSRV: High frequency estimation that is robust to sampling times," .