

CONSERVATIVE DELTA HEDGING¹

BY PER ASLAK MYKLAND

University of Chicago

It is common to have interval predictions for volatilities and other quantities governing securities prices. The purpose of this paper is to provide an exact method for converting such intervals into arbitrage based prices of financial derivatives or industrial or contractual options. We call this procedure *conservative delta hedging*. The proposed approach will permit an institution's management a greater oversight of its exposure to risk.

1. Introduction. How should one hedge an option when there is uncertainty about the probability distribution governing the underlying security? Apart from the possibility of substantial jumps in prices, this is perhaps the most major issue facing institutions that sell and hedge options.

It should be emphasized right away that there are, in fact, two such questions. Suppose that P is the true probability distribution of the underlying securities. One can now divide the problem into:

1. The “probabilistic problem”: P is fixed and known, but the “risk neutral probability” P^* [Harrison and Kreps (1979), Harrison and Pliskà (1981), Delbaen and Schachermayer (1994, 1995)] is unknown.
2. The “statistical problem”: P is not known.

The bulk of existing literature focuses on case (1). A broader formulation of this problem is to say that there is some form of incompleteness or other barrier to perfect hedging. This usually is the same as saying that P^* is unknown [though see, in particular, Delbaen and Schachermayer (1994, 1995)]. Strategies in such circumstances include super hedging [including Cvitanic (1998), Cvitanic and Karatzas (1992, 1993), Cvitanic, Pham and Touzi (1999a, b), El Karoui and Quenez (1995), Eberlein and Jacod (1997), Karatzas (1996), Karatzas and Kou (1996, 1998) and Kramkov (1996)]; mean variance hedging [Föllmer and Schweizer (1991), Föllmer and Sondermann (1986), Schweizer (1990, 1991, 1992, 1993, 1994) and later also Delbaen and Schachermayer (1996), Delbaen, Monat, Schachermayer, Schweizer and Stricker (1997), Laurent and Pham (1999) and Pham, Rheinländer, and Schweizer (1998)]; quantile style hedging [see, in particular, Külldorff (1993), Cvitanic (1998), Spivak and Cvitanic (1999) and Föllmer and Leukert (1999, 2000)]. There is

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also work on hedging in additional market traded securities, which can be done, for example, in the presence of a stochastic volatility model [see, e.g., Ball and Roma (1994), Hofmann, Platen and Schweizer (1992), Hull and White (1987), Pham and Touzi (1996) and Renault and Touzi (1996); see also Duffie (1996), Chapter 8.H].

In this paper, however, we shall confront question (2): what should one do when the probability distribution P of the underlying securities is unknown. Specifically, suppose one has a set \mathcal{P} of such probability distributions. Is there a way of hedging a given option so that the strategy at least finances the payoff no matter what the probability P is in \mathcal{P} ? In other words, how can one convert \mathcal{P} into a replicating strategy?

We shall show that such strategies exist, and that the general form is similar to that obtained for super hedging for given P (Sections 3, 5 and 6). We also show that the form of such strategies can be surprisingly simple, and we provide expressions for European options with convex payoffs (Sections 2 and 4). The relationship to the super hedging literature cited above is discussed just after Theorem 3.1.

A main contribution of our results is to provide a way of putting an interval representing uncertainty around the value of an institution's portfolio. The lack of such intervals would appear to be a serious problem, in that it reduces the effectiveness of management oversight and government regulation. Such issues seem to have been a major factor behind a number of recently reported substantial losses involving derivative securities. Headline-grabbing cases include those of the Bank of Tokyo-Mitsubishi (see, for example, article "Bank of Tokyo blames loss on bad model" in *The Wall Street Journal*, March 28, 1997, page A3), National Westminster Bank (*The Guardian*, April 23, 1997, Section 1, page 24), and the Union Bank of Switzerland (see *The Economist*, January 31, 1998, pages 18, and 75–77, including article, "Blind faith"). On top of that, there is the room which the lack of oversight gives to straightforward fraud.

An aspect of this problem is that accounting rules tend to try to mark the portfolio to market as much as possible. For illiquid instruments, this sometimes involves using prices arising from reported transactions where the price is quite wrong. Somebody had a bad day. Nonetheless, such a price is forced on others for accounting purposes. The methodology developed in this paper provides an alternative in this respect.

Previous literature on this subject has looked at the case where \mathcal{P} is the set of all diffusions for which the volatility falls within a certain interval at all times t . [Avellaneda, Levy and Paras (1995), and Lyons (1995); compare Example 4 in Section 3 of this paper. See also Bergman, Grundy and Wiener (1996), El Karoui, Jeanblanc-Picqué and Shreve (1998) and Hobson (1998), though the intent of these papers appears more to be about robustness than about prediction regions.] We shall see that the more general construction provided in this paper can provide substantially lower starting values for hedging strategies. Apart from this literature, confidence or prediction regions seem at this time to mainly be used in an ad hoc fashion. The value-at-risk literature

is an example of this [see Duffie and Pan (1997) for a review]. New work, however, by Cvitanić and Karatzas (1998) and Karatzas and Zhao (1998) seems to point toward putting value-at-risk on a more solid footing.

How is the set \mathcal{P} of probabilities given? It can be a confidence or Bayesian set, or it can be based on “expert judgement.” If one uses statistical methods, there is a substantial literature on inference for stochastic differential equations in finance. Highlights include Ait-Sahalia (1996, 1998), Andersen and Lund (1997), Bandi (1998), Bandi and Phillips (1998), Barndorff-Nielsen (1997), Barndorff-Nielsen and Shephard (1998), Bibby and Sørensen (1995, 1996a, b), Dachuna-Castelle and Florens-Zmirou (1986), Danielsson (1994), Florens-Zmirou (1993), Foster and Nelson (1996), Gallant and Tauchen (1997), Hansen and Scheinkman (1995), Hansen, Scheinkman, and Touzi (1998), Jacquier, Polson and Rossi (1994), Kessler and Sørensen (1999), Küchler and Sørensen (1998) and Lo (1987, 1988). For related discrete time processes, see Bollerslev, Chou and Kroner (1992) for an overview of ARCH and GARCH models.

In other words, this paper itself is not about statistical inference. Instead, it provides a way of converting the results of statistical inference into trading algorithms.

2. Conservative delta hedging. The approach we argue in this paper is the following: take a conservative stab at predicting the cumulative volatility and interest rate. Then hedge according to the (approximate) actual volatility and interest. This uses sharply the well-posed feature of our system—that one approximately observes the current interest rate and volatility. On the other hand, it guards against excessive reliance on the ill-posed one, namely the comparative lack of knowledge about P .

The basic idea, and the fact that it works, can be illustrated in the context of European call options. Suppose that a stock follows

$$(2.1) \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$$

and pays no dividends, and that there is a risk free interest rate r_t . Both r_t and σ_t can be stochastic and random. They, and S_t , have to be adapted to some underlying filtration, but it is not necessary to specify this filtration further. We are not saying, for example, how many “factors” there must be. This is similar to specifying a σ -field for one’s space of outcomes; it is there, but it is anonymous.

EXAMPLE 1. Consider a European call with strike price K that pays off at time T . Let

$$(2.2) \quad C(S, R, \Xi) = S\Phi(d_1) - K \exp(-R)\Phi(d_2),$$

where

$$(2.3) \quad d_1 = (\log(S/K) + R + \Xi/2)/\sqrt{\Xi}$$

and $d_2 = d_1 - \sqrt{\Xi}$. Note that if r and σ were fixed constants, then the Black–Scholes–Merton [Black and Scholes (1973), Merton (1973)] price of the option

at time t and stock price S would be $C(S, r(T-t), \sigma^2(T-t))$. We are, however, proposing to use the function (2.2) and (2.3) for more general purposes.

Consider the instrument whose value at time t is

$$(2.4) \quad V_t = C(S_t, R_t, \Xi_t),$$

where

$$(2.5) \quad R_t = R_0 - \int_0^t r_u du \quad \text{and} \quad \Xi_t = \Xi_0 - \int_0^t \sigma_u^2 du.$$

In equation (2.5), r_t and σ_t are the actual observed quantities. As mentioned above, they can be stochastic and random.

Now suppose that

$$(2.6) \quad R_0 \geq \int_0^T r_u du \quad \text{and} \quad \Xi_0 \geq \int_0^T \sigma_u^2 du.$$

By using Itô's formula, we shall see that no matter how r_t and σ_t actually otherwise behave (and whether they are random or nonrandom), there is a self-financing strategy for V_t , based on hedging in the security S_t and the money market bond

$$(2.7) \quad \beta_t = \exp\left(\int_0^t r_u du\right).$$

The reason why V_t is self financing, is that

$$(2.8) \quad \frac{1}{2}C_{SS}S^2 = C_{\Xi} \quad \text{and} \quad -C_R = C - C_S S.$$

This can be obtained directly by differentiating (2.2). Also, the first of the two equations in (2.8) is the well-known relationship between the "gamma" and the "vega" [cf., for example, Chapter 14 of Hull (1997)].

Hence, by Itô's Lemma, dV_t equals

$$(2.9) \quad \begin{aligned} dC(S_t, \Xi_t, R_t) &= C_S dS_t + \frac{1}{2}C_{SS}S_t^2 \sigma_t^2 dt + C_{\Xi} d\Xi_t + C_R dR_t \\ &= C_S dS_t + (C - C_S S_t)r_t dt + [\frac{1}{2}C_{SS}S_t^2 - C_{\Xi}]\sigma_t^2 dt \\ &\quad + [-C + C_S S_t - C_R]r_t dt. \end{aligned}$$

In view of (2.8), the last two lines of (2.9) vanish, and hence there is a self-financing hedging strategy for V_t in S_t and β_t . The "delta" (the number of stocks held) is $C'_S(S_t, R_t, \Xi_t)$.

Furthermore, since $C(S, \Xi, R)$ is increasing in Ξ and R , (2.6) yields that

$$(2.10) \quad \begin{aligned} V_T &= C(S_T, \Xi_T, R_T) \\ &\geq \lim_{\Xi \downarrow 0, R \downarrow 0} C(S_T, \Xi, R) \\ &= (S_T - K)^+ \end{aligned}$$

almost surely. In other words, one can both synthetically create the security V_t , and one can use this security to cover one's obligations.

The above is sufficiently innocent looking that it should be emphasized that it is not trivial. Keep in mind that it is true for *all* probability distributions under which (2.6) is satisfied with probability 1. We again emphasize that this includes stochastic volatility and stochastic interest rates.

Note that a lot of this example carries over to general European options. In particular, for payoff $f(S_T)$, one can write the Black–Scholes–Merton price of the option as

$$(2.11) \quad C(S, \Xi, R) = \exp(-R) E f\left(S \exp(R - \Xi/2 + \sqrt{\Xi} Z)\right),$$

where Z is standard normal [see, for example, Chapter 6 of Duffie (1996)]. Hence (2.8) follows by differentiation for all European options. For noncall options, however, it may be more suitable to specify intervals in a different manner to assure (2.10).

Unless the cumulative interest is actually known beforehand, however, the scheme given above is not quite right. This is in the sense that a lower price can usually be found as the starting point for a self-financing strategy satisfying (2.10). The following is an example of this.

EXAMPLE 2. What the procedure in Example 1 overlooks, is that the price Λ_0 of the zero coupon bond maturing at T is actually known (at least in most markets). This bond satisfies

$$(2.12) \quad \Lambda_t = E^* \left[\exp\left(-\int_t^T r_u du\right) \middle| \mathcal{F}_t \right],$$

where P^* is the risk-neutral distribution, and the procedure we have given ignores that. A consequence, for example, is that if one prices a forward contract the same way, one gets the price wrong.

It will sometimes be the case that the existence of security Λ_t will also make lower bounds for r and σ active, and one can consider replacing (2.6) by

$$(2.13) \quad R^+ \geq \int_0^T r_u du \geq R^- \quad \text{and} \quad \Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-.$$

We shall see in Section 4 how to compute the value of the call option in this case. We show that if v_0 solves

$$(2.14) \quad \Phi(d_2(S_0, v_0, \Xi^+)) = \frac{\exp(-R^-) - \Lambda_0}{\exp(-R^-) - \exp(-R^+)},$$

where Φ is the cumulative normal distribution and in the same notation as in (2.2) and (2.3), then one can start a super-replicating strategy with the price at time zero given in the following:

$$(2.15) \quad \begin{aligned} v_0 \geq R^+ &: C(S_0, R^+, \Xi^+) \\ R^+ > v_0 > R^- &: C(S_0, v_0, \Xi^+) \\ &+ K(\exp(-v_0) - \exp(-R^+))\Phi(d_2(S_0, v_0, \Xi^+)) \\ v_0 \leq R^- &: C(S_0, R^-, \Xi^+) + K(\exp(-R^-) - \Lambda_0). \end{aligned}$$

An illustration of the improvement over Example 1 is given in Figure 1.

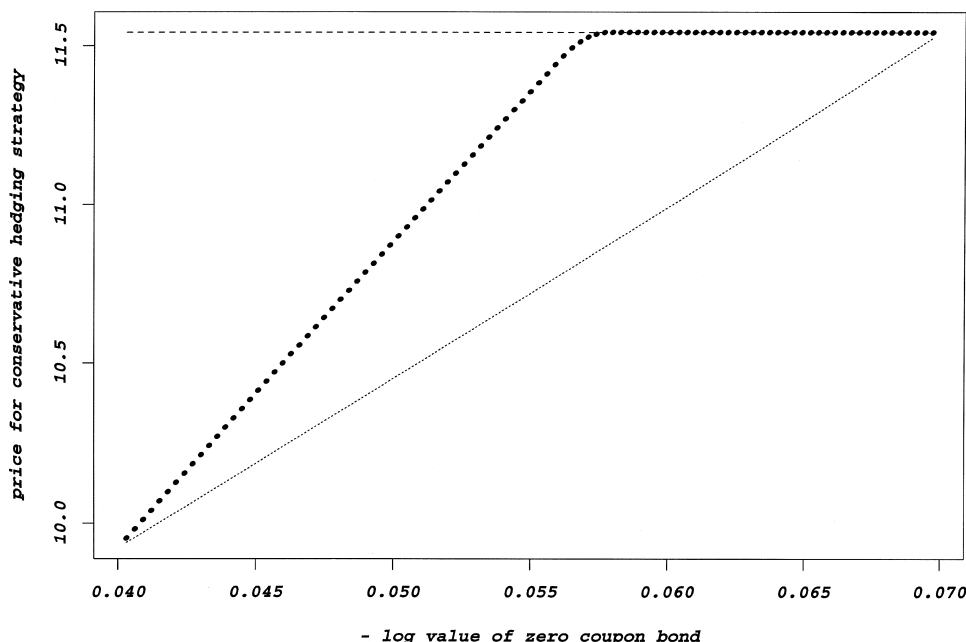


FIG. 1. Option price as $-\log(\Lambda_0)$ varies between R^- and R^+ . Three ways of pricing European call options in the presence of restrictions (2.13). The upper dashed line is based on the strategy from Example 1, while the heavily dotted line uses the strategy from Example 2. The almost diagonal lower line is the plug-in price $C(S_0, -\log \Lambda_0, \Xi^+)$. The latter cannot be used as the starting value of a conservative hedging strategy, but is included for reference. Parameter values are $\Xi^+ = 0.04$, $R^- = 0.04$ and $R^+ = 0.07$. The option is at the money ($K = S_0$).

In the presence of market traded derivatives, one can further bring down the starting value of the trading strategy by hedging in these in addition to the zero coupon bond.

3. General theory. Consider a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t)_{0 \leq t \leq T}$, and adapted continuous processes $S_t^{(1)}, \dots, S_t^{(p)}$, representing traded securities paying no dividends. Here r_t is an adapted processes representing the risk-free interest rate, and $\beta_t = \exp\{\int_0^t r_u du\}$ is the value at time t of one dollar deposited in the money market at time 0. \mathcal{P} is a set of probability distributions on (Ω, \mathcal{F}) .

We find ourselves in the following situation. We stand at time $t = 0$, and we have to make a payoff η at a (nonrandom or stopping) time τ ; η is \mathcal{F}_τ -measurable. We do not know what the probability distribution for this system is, but we know that it is an element in \mathcal{P} . We are looking for hedging strategies in $S_t^{(1)}, \dots, S_t^{(p)}, \beta_t$ that will superreplicate the payoff with probability 1.

DEFINITION. A property will be said to hold \mathcal{P} -a.s. if it holds P -a.s. for all $P \in \mathcal{P}$. For any process X_t , $X_t^* = \beta^{-1}X_t$, and vice versa.

A process $V_t, 0 \leq t \leq T$, is said to be a superreplication of payoff η provided V_t is adapted, cadlag and provided:

- (i) One can cover one's obligations,

$$(3.1) \quad V_\tau \geq \eta$$

\mathcal{P} -a.s.

- (ii) For all $P \in \mathcal{P}$, there are processes H_t and D_t , so that, for all $t, 0 \leq t \leq T$,

$$(3.2) \quad V_t = H_t + D_t, \quad 0 \leq t \leq T,$$

where D_t^* is a nonincreasing process and where H_t is self financing in the traded securities $S_t^{(1)}, \dots, S_t^{(p)}$.

“Self financing” means, by numeraire invariance [see, for example, Section 6.B of Duffie (1996)], that H_t^* can be represented as a stochastic integral with respect to the $S_t^{(i)*}$ s, subject to regularity conditions to eliminate doubling strategies. There is some variation in how to implement this [see, e.g., Duffie (1996), Chapter 6.C, (pages 103–105)]. For this reason, the precise definition is deferred until after the main theorem and some examples (see Section 5).

What should the price be for a promise of a payoff η ? A natural hedging based approach is as follows [cf. Cvitanić and Karatzas (1992, 1993), El Karoui and Quenez (1995) and Kramkov (1996)].

DEFINITION. The *conservative ask price* (or offer price) at time 0 for a payoff η to be made at a time τ is

$$(3.3) \quad A = \inf \{V_0: (V_t) \text{ is a superreplication of the payoff}\}.$$

Similarly, the *conservative bid price* can be defined as the supremum over all subreplications of the payoff, in the obvious sense.

The conservative bid price B equals

$$(3.4) \quad B(\eta) = -A(-\eta)$$

in obvious notation, and subject to mild regularity conditions. For this reason, it is enough to study ask prices. More generally, if one already has a portfolio of options, one may wish to charge $A(\text{portfolio} + \eta) - A(\text{portfolio})$ for the payoff η .

To give the general form of the ask price A , we consider an appropriate set \mathcal{P}^* of “risk neutral” probability distributions P^* .

DEFINITION. Set

$$(3.5) \quad \mathcal{N} = \{C \subseteq \Omega: \forall P \in \mathcal{P} \exists E \in \mathcal{F}: C \subseteq E \text{ and } P(E) = 0\}.$$

\mathcal{P}^* is now defined as the set of probability measures P^* on \mathcal{F} whose null sets include those in \mathcal{N} and for which $S_t^{(1)*}, \dots, S_t^{(p)*}$ are martingales. We

also define \mathcal{P}^e as the set of extremal elements in \mathcal{P}^* . P^e is extremal in \mathcal{P}^* if $P^e \in \mathcal{P}^*$ and if, whenever $P^e = a_1 P_1^e + a_2 P_2^e$ for $a_1, a_2 > 0$ and $P_1^e, P_2^e \in \mathcal{P}^*$, it must be the case that $P^e = P_1^e = P_2^e$.

Subject to regularity conditions, we shall show that there is a superreplicating strategy V_t with initial value A , and characterize the set of such strategies. Under weak additional assumptions, it will be the case that

$$(3.6) \quad A = \sup\{E^*(\eta^*): P^* \in \mathcal{P}^*\},$$

where

$$(3.7) \quad \eta^* = \exp\left\{-\int_0^T r_u du\right\}\eta.$$

The result (3.6) hinges on the existence of the “essential suprema”

$$(3.8) \quad A_t^* = \text{ess sup}\{E^*(\eta^* | \mathcal{F}_t): P^* \in \mathcal{P}^*\}.$$

Equation (3.8) means that A_t^* is (in a suitable sense) the smallest measurable random variable so that $A_t^* \geq E^*(\eta^* | \mathcal{F}_t)$ for all $P^* \in \mathcal{P}^*$. An exact definition of this object is given in the Appendix. We can then state the following theorem.

THEOREM 3.1. *Assume the following conditions on the system: (\mathcal{F}_t) is right continuous; \mathcal{F}_0 is the smallest σ -field containing \mathcal{N} ; the $S_t^{(i)}$ are \mathcal{P} -a.s. continuous and adapted; the short rate process r_t is adapted, and integrable \mathcal{P} -a.s.; every $P \in \mathcal{P}$ has an equivalent martingale measure, that is to say that there is a $P^* \in \mathcal{P}^*$ that is equivalent to P . Also assume the following conditions on the payoff: $0 \leq \tau \leq T$, where T is nonrandom and finite, η is \mathcal{F}_τ -measurable, and*

$$\sup_{P^* \in \mathcal{P}^*} E^*|\eta^*| < \infty.$$

Define the following conditions. (E_1) : If X is a bounded random variable and there is a $P^ \in \mathcal{P}^*$ so that $E^*(X) > 0$, then there is a $P^e \in \mathcal{P}^e$ so that $E^e(X) > 0$. (E_2) : There is a real number K so that $\{\eta^* \geq K\}^c \in \mathcal{N}$. Let A be given by (3.3).*

(i) *Suppose that (V_t) is an adapted process satisfying (3.1). Assume either condition (E_1) or (E_2) , or that (V_t) is continuous. Then (V_t) is a superreplication of η if and only if (V_t^*) is a cadlag supermartingale for all $P^* \in \mathcal{P}^*$.*

(ii) *Suppose that a superreplication of η exists, and assume condition (E_1) or condition (E_2) . Then there is a superreplication so that $V_0 = A$.*

(iii) *Assume that the right-hand side of (3.8) is well defined for all $t, 0 \leq t \leq T$. Assume either condition (E_1) or (E_2) or that (A_t) can be taken to be continuous. Then (A_t) has a right continuous modification which is a superreplication of η , $A_0 = A$, and A is given by (3.6).*

Note that under condition (E_2) , Theorem 3.1(i) is a corollary to Theorem 2.1 (p. 461) of Kramkov (1996). This is because \mathcal{P}^* includes the union of the equivalent martingale measures of the elements in \mathcal{P} . For reasons of symmetry, however, we have also sought to study the case where η^* is not bounded below,

whence the condition (E_1) . The need for symmetry arises from the desire to also study bid prices; compare (3.4). For example, neither a short call nor a short put are bounded below. See Section 5.

Also note that the main difference between this paper and the literature on superhedging for given P [such as Kramkov (1996) and other papers cited in the introduction] is that we have to deal with things like essential suprema in the undominated case. These do not automatically exist; see also the development in the Appendix. This structures the conditions, definitions and results in this section.

A requirement in the above theorem that does need some comment is the one involving extremal probabilities. Condition (E_1) is actually quite weak, as it is satisfied when \mathcal{P}^* is the convex hull of its extremal points. Sufficient conditions for a result of this type are given in Theorems 15.2, 15.3 and 15.12, pages 496–498, in Jacod (1979). For example, the first of these results gives the following as a special case (see Section 6). This will cover our examples.

PROPOSITION 3.2. *Assume the conditions of Theorem 3.1. Suppose that r_t is bounded below by a nonrandom constant (greater than $-\infty$). Suppose that (\mathcal{F}_t) is the smallest right continuous filtration for which $(\beta_t, S_t^{(1)}, \dots, S_t^{(p)})$ is adapted and so that $\mathcal{N} \subseteq \mathcal{F}_0$. Let $C \in \mathcal{F}_T$. Suppose that \mathcal{P}^* equals the set of all probabilities P^* so that $(S_t^{(1)*}), \dots, (S_t^{(p)*})$ are P^* -martingales, and so that $P^*(C) = 1$. Then Condition (E_1) is satisfied.*

EXAMPLE 3. To see how the above works, consider systems with only one stock ($p = 1$). We let (β_t, S_t) generate (\mathcal{F}_t) . A set $C \in \mathcal{F}_T$ will describe our restrictions. For example, C can be the set given by (2.6), or (2.13), or (3.11) below. The fact that σ_t is only defined given a probability distribution is not a difficulty here: we consider P 's so that the set C has probability 1 (where quantities like σ_t are defined under P).

One can also work with other types of restrictions. For example, C can be the set of probabilities so that (2.13) is satisfied, and also $\Pi^- \leq [r, \sigma]_T \leq \Pi^+$. Only the imagination is the limit here. Of course, as we shall argue in Example 4, some “prediction” sets are better than others, and determining such good sets is an interesting question for further research.

Hence, \mathcal{P} is the set of all probability distributions P so that $S_0 = s_0$ (the actual value),

$$(3.9) \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t,$$

with r_t integrable P -a.s., and bounded below by a nonrandom constant, so that $P(C) = 1$, and so that

$$(3.10) \quad \exp \left\{ - \int_0^t \lambda_u dW_u - \frac{1}{2} \int_0^t \lambda_u^2 du \right\} \text{ is a } P\text{-martingale,}$$

where $\lambda_u = (\mu_u - r_u)/\sigma_u$. The condition (3.10) is what one needs for Girsanov's theorem [see, for example, Karatzas and Shreve (1991), Theorem 3.5.1] to hold,

which is what assures the required existence of an equivalent martingale measure. Hence, in view of Proposition 3.2, Condition (E₁) in Theorem 3.1 is taken care of.

To gain more flexibility, one can let (\mathcal{F}_t) be generated by more than one stock, and just let these stocks remain “anonymous.” One can then still use condition (E₁). Alternatively, if the payoff is bounded below, one can use condition (E₂).

EXAMPLE 1 (Continued from Section 2). This can fall under Example 3 above, by taking C to be the set where (2.6) holds. In this case, however, the problem is already solved in Section 2 by guessing the form of A_t and then applying Itô’s lemma. The reason why this is the least expensive strategy is that the stated upper bound coincides with the Black–Scholes–Merton (1973) price for constant coefficients $r = R_0/T$ and $\sigma^2 = \Xi_0/T$. This is one possible realization satisfying the constraint (2.6). Theorem 3.1, however, gives a systematic way of arriving at the same result.

EXAMPLE 2 (European call; hedging in the stock and a zero coupon bond. Continued from Section 2). Here, $p = 2$, $S_t^{(1)} = S_t$, $S_t^{(2)} = \Lambda_t$. This does not fall under Example 3. Further details are explored in Section 4.

EXAMPLE 4 [Avellaneda, Levy, and Paras (1995)]. The setup can be covered by Example 3, but now the interest rate r is constant and nonrandom, and C is the set for which

$$(3.11) \quad \sigma_t \in [\sigma^-, \sigma^+] \quad \text{for all } t \in [0, T].$$

To make the set C -measurable, one can suppose, for example, that σ_t is cadlag. With this assumption, a superreplicating strategy is constructed for European options based on the Black–Scholes–Barenblatt equation [cf. Barenblatt (1979)].

For the same reasons as in Example 1, the price given by Avellaneda, Levy and Paras (1995) is the conservative ask price. Note, however, that the hedging strategy presented by that paper is not the one used here. Instead of basing themselves on the actually accrued volatility [as in (2.5)], they hedge based on a worst case nonobserved volatility. The reason why they can get away with this is that the method of specifying conservativeness is particularly conservative; if one supposes a volatility model, say,

$$(3.12) \quad d\sigma_t = \nu(\sigma_t) dt + \gamma(\sigma_t) dB_t,$$

a 95% (say) bound of the type (3.11) yields a much higher price than one of the type (2.6), since the latter involves the average of σ_t^2 while the former depends on the maximum and the minimum of this process. The high prices that are inherent in the Avellaneda, Levy and Paras (1995) approach has been commented on by Ahn, Muni and Swindle (1996a, b), who suggest ways of alleviating the effect.

4. A case study: convex European options hedged in a stock and a zero coupon bond. As in Example 2 (see Sections 2 and 3), $p = 2$, $S_t^{(1)} = S_t$, $S_t^{(2)} = \Lambda_t$. \mathcal{P} is assumed to be the set of probability measures where:

1. S_t and Λ_t are continuous and adapted semimartingales, with $S_0 = s_0$ and $\Lambda_0 = \lambda_0$ (the actual values); r_t is adapted, integrable and bounded below by a nonrandom constant.
2. $[\log S, \log S]_t$ is absolutely continuous (w.r.t. Lebesgue measure), with derivative σ_t^2 .
3. (2.13) holds.
4. Every $P \in \mathcal{P}$ has an equivalent martingale measure P^* , that is, $P^* \sim P$, so that S_t^* and Λ_t^* are P^* -martingales.

In view of Girsanov’s theorem, the last condition is tantamount to imposing conditions along the lines of what we did with Example 1 in Section 2.

We consider ask prices of options with payoff $\eta = f(S_T)$ at (nonrandom) time T , where f is convex. This includes calls and puts. For convenience, we suppose that f is bounded below. Hence can we invoke condition (E_2) of Theorem 3.1.

First, as far as (2.13) is concerned, Ξ^+ is attained. This is because, if not, one can rescale time $\tilde{r}_t = r_{t/(1-\varepsilon)}$, $\tilde{S}_t = S_{t/(1-\varepsilon)}$ for $0 \leq t \leq T(1-\varepsilon)$, with $\tilde{r}_t = 0$ for $(1-\varepsilon)T \leq t \leq T$ and let \tilde{S}_t run up the full unused volatility on the same interval, say $\tilde{\sigma}_t^2 = (\Xi^+ - \Xi_{T-\varepsilon})/\varepsilon$. In other words, suppose that $\tilde{A}_0 = E^* \exp\{-\int_0^T r_u du\}f(S_T)$ is within δ of the maximum. Then rewrite $\tilde{A}_0 = E^* \exp\{-\int_0^{T-\varepsilon} \tilde{r}_u du\}f(\tilde{S}_{T-\varepsilon})$. If we prolong \tilde{r}_t and $\tilde{\sigma}_t^2$ as indicated on the time interval, we get $E^* \exp\{-\int_0^T \tilde{r}_u du\}f(\tilde{S}_T) \geq \tilde{A}_0$. This is because \tilde{S}_t^* is a martingale, whence it follows that $\exp(-\int_0^t \tilde{r}_u du)f(\tilde{S}_t)$ is a submartingale from $(1-\varepsilon)T$ to T , since \tilde{r}_u is zero on this time interval.

In consequence, the conservative ask price at time 0 is in this case

$$(4.1) \quad A_0 = \sup_X E^* X f(S_T^*/X),$$

where $\log S_T^* - \log S_0^*$ is normal $N(-\frac{1}{2}\Xi^+, \Xi^+)$, and the supremum is over all random variables X for which

$$(4.2) \quad \exp(-R^+) \leq X \leq \exp(-R^-)$$

and satisfying

$$(4.3) \quad E^* X = \Lambda_0.$$

This is provided we can show that, for the relevant P^* , there is a β_t and a Λ_t so that $\beta_T = X^{-1}$.

Since $x \rightarrow xf(s/x)$ is convex (for $x, s > 0$), the functional $X \rightarrow E^* X f(S_T^*/X)$ is also convex, and so the supremum in (4.1) is attained for an X which only takes the values $\exp(-R^-)$ and $\exp(-R^+)$.

Let $B^\pm = \{\omega: X(\omega) = \exp(-R^\pm)\}$ for $\pm = +$ or $-$. It follows from the above that we wish to maximize

$$(4.4) \quad E^* \exp(-R^+)f(\exp(R^+)S_T^*)I_{B^+} + E^* \exp(-R^-)f(\exp(R^-)S_T^*)I_{B^-}$$

subject to

$$(4.5) \quad \exp(-R^+)P^*(B^+) + \exp(-R^-)P^*(B^-) = \Lambda_0.$$

Now let

$$(4.6) \quad B^+ = \{\omega: S_T^* \geq \tilde{K}\},$$

where \tilde{K} is determined by (4.5), and let \tilde{B}^\pm be any other pair of sets satisfying (4.5). Since f is convex, $s \rightarrow \exp(-R^-)f(\exp(R^-)s) - \exp(-R^+)f(\exp(R^+)s)$ is a nonincreasing function. Hence,

$$\begin{aligned} & \exp(-R^+)f(\exp(R^+)S_T^*)(I_{B^+} - I_{\tilde{B}^+}) + \exp(-R^-)f(\exp(R^-)S_T^*)(I_{B^-} - I_{\tilde{B}^-}) \\ & \geq (\exp(-R^+)f(\exp(R^+)\tilde{K}) - \exp(-R^-)f(\exp(R^-)\tilde{K}))(I_{B^+} - I_{\tilde{B}^+}), \end{aligned}$$

and so the choice (4.6) maximizes (4.4) subject to (4.5). In the notation (2.2) and (2.3), \tilde{K} is found from (4.5) by

$$(4.7) \quad \Phi(d_2(S_0, \tilde{K}, 0, \Xi^+)) = \frac{\exp(-R^-) - \Lambda_0}{\exp(-R^-) - \exp(-R^+)}.$$

Note that this equation always has a (unique) solution because $\exp(-R^-) \geq \Lambda_0 \geq \exp(-R^+)$.

To see that the maximum is indeed the ask price, we now construct a $P^* \in \mathcal{P}^*$ as follows. Fix t_0 so that $0 < t_0 < T$. On $[0, t_0]$, we let $dS_t^* = \sigma S_t^* dW_t$, with $\sigma^2 = \Xi^+/t_0$, and we set $r_t = R^-/t_0$. From t_0 to T , S_t^* is constant ($= S_{t_0}^*$). If $S_{t_0}^* < \tilde{K}$, then $r_t = 0$; otherwise $r_t = (R^+ - R^-)/(T - t_0)$. This gives us an allowable β_t , and Λ_t^* is obviously

$$(4.8) \quad \Lambda_t^* = \exp(-R^+)P(B^+ | S_t) + \exp(-R^-)P(B^- | S_t).$$

The above gives the maximum for $t = 0$, but the same argument can be carried out for all t , $0 \leq t \leq T$, thereby also assuring the existence of the essential supremum. We have thus proved the following proposition.

PROPOSITION 4.1. *Subject to the assumptions at the beginning of Section 4, the conservative ask price at time zero for payoff $f(S_T)$ at time T is*

$$(4.9) \quad \begin{aligned} A = & E^* \exp(-R^+)f(\exp(R^+)S)I_{\{S \geq \tilde{K}\}} \\ & + E^* \exp(-R^-)f(\exp(R^-)S)I_{\{S < \tilde{K}\}}, \end{aligned}$$

where \tilde{K} is given by (4.7), and where $\log S$ is normal $N(\log S_0 - \frac{1}{2}\Xi^+, \Xi^+)$.

Alternatively, one can obviously write

$$(4.10) \quad \begin{aligned} A &= C(S_0, R^+, \Xi^+, f(s)I(s \geq \tilde{K} \exp(R^+)) \\ &\quad + C(S_0, R^-, \Xi^+, f(s)I(s < \tilde{K} \exp(R^+)), \end{aligned}$$

where $C(S_0, R, \Xi, h(s))$ is the Black–Scholes–Merton price for payoff $h(S_T)$ at time T .

Note that if one wishes to disregard the bond Λ_t (as in Example 1 in the preceding sections), one just maximizes (4.4) without (4.5).

Now for the European call.

EXAMPLE 2 (Continued from Sections 2 and 3). If K is the strike price, $f(s) = (s - K)^+$. The solution given in (2.14) and (2.15) follows directly by setting $v_0 = \log(K/\tilde{K})$.

5. Defining self-financing strategies. In essence, H_t being self financing means that we can represent H_t^* by

$$(5.1) \quad H_t^* = H_0^* + \sum_{i=1}^p \int_0^t \theta_s^{(i)} dS_s^{(i)*}.$$

This is in view of numeraire invariance [see, e.g., Section 6.B of Duffie (1996)].

Fix $P \in \mathcal{P}$, and recall that the $S_t^{(i)*}$ are continuous. We shall take the stochastic integral to be defined when $\theta_t^{(1)}, \dots, \theta_t^{(p)}$ is an element in $L^2_{loc}(P)$, which is the set of p -dimensional predictable processes so that $\int_0^t \theta_u^{(i)2} d[S^{(i)*}, S^{(i)*}]_u$ is locally integrable P -a.s. The stochastic integral (5.1) is then defined by the process in Theorems I.4.31 and I.4.40, (pages 46–48), in Jacod and Shiryaev (1987).

A restriction is needed to be able to rule out doubling strategies. The two most popular ways of doing that are to insist either that H_t^* be in an L^2 -space, or that it be bounded below [Harrison and Kreps (1979), Delbaen and Schachermayer (1995), Dybvig and Huang (1988), Karatzas (1996); see also Duffie (1996), Section 6.C]. We shall here go with a criterion that encompasses both.

DEFINITION. A process $H_t, 0 \leq t \leq T$, is *self financing* with respect to $S_t^{(1)}, \dots, S_t^{(p)}$ if H_t^* satisfies (5.1), and if $\{H_\lambda^{*-}, 0 \leq \lambda \leq T, \lambda \text{ stopping time}\}$ is uniformly integrable under all $P^* \in \mathcal{P}^*$ that are equivalent to P .

The reason for seeking to avoid the requirement that H_t^* be bounded below is that, to the extent possible, the same theory should apply equally to bid and ask prices. Since the bid price is normally given by (3.4), securities that are unbounded below will be a common phenomenon. For example, $B((S - K)^+) = -A(-(S - K)^+)$, and $-(S - K)^+$ is unbounded below.

It should be emphasized that our definition does, indeed, preclude doubling type strategies. The following is a direct consequence of optional stopping and Fatou’s lemma.

PROPOSITION 5.1. *Let $P \in \mathcal{P}$, and suppose that there is at least one $P^* \in \mathcal{P}^*$ that is equivalent to P . Suppose that H_t^* is self financing in the sense given above. Then, if there are stopping times λ and μ , $0 \leq \lambda \leq \mu \leq T$, so that $H_\mu^* \geq H_\lambda^*$, P -a.s., then $H_\mu^* = H_\lambda^*$, P -a.s.*

Note that Proposition 5.1 is, in a sense, an equivalence. If the conclusion holds for all H_t^* , it must in particular hold for those that Delbaen and Schachermayer (1995) term admissible. Hence, by Theorem 1.4, page 929, of their work, P^* exists.

6. Proofs for Section 3.

PROOF OF THEOREM 3.1(i). The “only if” part of the result is obvious, so it remains to show the “if” part.

(a) *Structure of the Doob–Meyer decomposition of (V_t^*) .* Fix $P^* \in \mathcal{P}^*$. Let

$$(6.1) \quad V_t^* = H_t^* + D_t^*, \quad D_0 = 0$$

be the Doob–Meyer decomposition of V_t^* under this distribution. The decomposition is valid by, for example, Theorem 8.22, page 83, in Elliot (1982). Then $\{H_\lambda^*, 0 \leq \lambda \leq T, \lambda \text{ stopping time}\}$ is uniformly integrable under P^* . This is because $H_t^{*-} \leq V_t^{*-} \leq E^*(|\eta^*| \mid \mathcal{F}_t^-)$, the latter inequality because $V_t^{*-} = (-V_t^*)^+$, which is a submartingale since V_t^* is a supermartingale. Hence uniform integrability follows by, say, Theorem I.1.42(b), page 11, of Jacod and Shiryaev (1987).

(b) Under condition (E_1) , (V_t) can be written $V_t^* = V_t^{*c} + V_t^{*d}$, where (V_t^{*c}) is a continuous supermartingale for all $P^* \in \mathcal{P}^*$, and (V_t^{*d}) is a nonincreasing process. Consider the set C of $\omega \in \Omega$ so that $\Delta V_t^* \leq 0$ for all t , and so that $V_t^{*d} = \sum_{s \leq t} \Delta V_s^*$ is well defined. We want to show that the complement $C^c \in \mathcal{N}$. To this end, invoke condition (E_1) , which means that we only have to prove that $P^e(C) = 1$ for all $P^e \in \mathcal{P}^e$.

Fix, therefore, $P^e \in \mathcal{P}^e$, and let H_t^* and D_t^* be given by the Doob–Meyer decomposition (6.1) under this distribution. By Proposition 11.14, page 345, in Jacod (1979), P^e is extremal in the set $M(\{S^{(1)*}, \dots, S^{(D)*}\})$ (in Jacod’s notation), and so it follows from Theorem 11.2, page 338, in the same work, that (H_t^*) can be represented as a stochastic integral over the $(S_t^{(i)*})$ ’s, whence (H_t^*) is continuous. $P^e(C) = 1$ follows.

To see that (V_t^{*c}) is a supermartingale for any given $P^* \in \mathcal{P}^*$, note that condition (E_1) again means that we only have to prove this for all $P^e \in \mathcal{P}^e$. The latter, however, follows from the decomposition in the previous paragraph.

(b) follows.

(c) (V_t^*) is a superreplication of η . Under condition (E_2) , the result follows directly from Theorem 2.1, page 461, of Kramkov (1996). Under the other conditions stated, by (b) above, one can take (V_t^*) to be continuous without losing generality. Hence, by local boundedness, the result also in this case follows from the cited theorem of Kramkov. \square

PROOF OF THEOREM 3.1(ii). Let $(V_t^{(n)})$ be a superreplication satisfying $V_0^{(n)} \leq A + 1/n$. Set $V_t = \inf_n V_t^{(n)}$. (V_t) is a supermartingale for all $P^* \in \mathcal{P}^*$. By Proposition 1.3.14, page 16, in Karatzas and Shreve (1991), (V_{t+}^*) (taken as a limit through rationals) exists and is a cadlag supermartingale except on a set in \mathcal{N} . Hence (V_{t+}^*) is a superreplication of η , with initial value no greater than A . The result follows from Theorem 3.1(i). \square

PROOF OF THEOREM 3.1(iii). Let A_t^* satisfy (3.8). This process is then a supermartingale for all $P^* \in \mathcal{P}^*$, by Proposition A.1 in the Appendix. If one forms (A_{t+}^*) as in the proof of Theorem 3.1(ii) above, however, $A_{t+}^* \leq A_t$ \mathcal{P}^* -a.s. Since, however, for all $P^* \in \mathcal{P}^*$, $A_{t+}^* \geq E^*(\eta^* | \mathcal{F}_t)$, A_{t+}^* must equal A_t \mathcal{P}^* -a.s. Since, obviously, the right-hand side in (3.6) is a lower bound for A , the result follows from Theorem 3.1(i). \square

PROOF OF PROPOSITION 3.2. Suppose that $r_t \geq -c$ for some $c < \infty$. We use Theorem 15.2c, page 496, in Jacod (1979). This theorem requires the notation $Ss^1(X)$, which is the set of probabilities under which the process X_t is indistinguishable from a submartingale so that $E \sup_{0 \leq s \leq t} |X_s| < \infty$ for all t (in our case, t is bounded, so things simplify). [cf. pages 353 and 356 of Jacod (1979).]

Jacod's result 15.2c studies, among other things, the set (in Jacod's notation) $S = \bigcap_{X \in \mathcal{X}} Ss^1(X)$, and under conditions which are satisfied if we take \mathcal{X} to consist of our processes $S_t^{(1)*}, \dots, S_t^{(p)*}, -S_t^{(1)*}, \dots, -S_t^{(p)*}, \beta_t e^{ct}, Y_t$. Here, $Y_t = 1$ for $t < T$, and I_C for $t = T$. (If necessary, $\beta_t e^{ct}$ can be localized to be bounded, which makes things messier but yields the same result). In other words, S is the set of probability distributions so that the $S_t^{(1)*}, \dots, S_t^{(p)*}$ are martingales, r_t is bounded below by c , and the probability of C is 1.

Theorem 15.2(c) now asserts a representation of all the elements in the set S in terms of its extremal points. In particular, any set that has probability zero for the extremal elements of S also has probability zero for all other elements of S .

However, $S = \widetilde{M}(\{S^{(1)*}, \dots, S^{(p)*}\})$ (again in Jacod's notation; see page 345 of that work); this is the set of extremal probabilities among those making $S^{(1)*}, \dots, S^{(p)*}$ a martingale. Hence, our condition (E_1) is proved. \square

7. Further questions. The above has provided a general methodology for converting sets of probabilities into prices and hedging strategies. Many questions of implementation remain, however, in the realm of future research. Can one substantially lower the price of the super hedge by choosing an optimal prediction region? Or by hedging in additional market traded securities? How does one compute ask and bid prices for more exotic derivatives than convex European payoffs? What about dynamically adjusted contingency reserves?

On the more statistical front, it is also worth pursuing a stronger result than our Theorem 3.2, as follows. If one has a prediction interval of the form (2.6), and one uses conservative delta hedging as described, the probability of failure is the same as the probability of the prediction interval not covering the

realized values. It remains unsolved whether this is the case in the general setting of Theorem 3.1.

There are a lot of unanswered questions.

APPENDIX

We here deal with the measure-theoretic details having to do with the essential supremum of conditional expectations, in the (apparent) absence of a dominating probability distribution. If such a distribution were to exist, one could, for example, use the result in Proposition VI-1-1, page 121, of Neveu (1975). In our case, we shall define the relevant objects in analogy to Neveu, but we need stronger assumptions to pin down the properties needed. There may be other ways of tackling the measurability problems; see, for example, Srivastava (1998).

DEFINITION. Suppose that \mathcal{D} is a collection of probability distributions on \mathcal{F} , and let \mathcal{G} be a sub- σ -field of \mathcal{F} . Let X be a random variable satisfying

$$(A.1) \quad \sup_{P \in \mathcal{D}} E|X| < \infty.$$

(a) If \mathcal{D} is countable, then $\text{ess sup}_{P \in \mathcal{D}} E(X | \mathcal{G})$ is the random variable Z for which:

- (i) Z is \mathcal{G} -measurable.
- (ii) $Z \geq E(X | \mathcal{G})$, P -a.s., for all $P \in \mathcal{D}$.
- (iii) If \tilde{Z} also satisfies (i) and (ii), then $\tilde{Z} \geq Z$, P -a.s., for all $P \in \mathcal{D}$.

(b) For general \mathcal{D} , $\text{ess sup}_{P \in \mathcal{D}} E(X | \mathcal{G})$ is the random variable Z for which (i) and (ii) hold, and satisfying

(iii') For all $P \in \mathcal{D}$, there is a subset \mathcal{D}' of \mathcal{D} , \mathcal{D}' countable, $P \in \mathcal{D}'$, so that $Z = \text{ess sup}_{R \in \mathcal{D}'} E_R(X | \mathcal{G})$ P -a.s.

Provided this quantity exists, it is, obviously, unique up to joint null sets of \mathcal{D} . Also, (iii') implies (iii). If \mathcal{D} is undominated in terms of absolute continuity, however, existence does not appear to be assured.

PROPOSITION A.1. Assume the conditions of Theorem 3.1. Also assume that A_t^* , given by (3.8), is defined. Then A_t^* is an (\mathcal{F}_t) -supermartingale for all $P^* \in \mathcal{P}^*$.

PROOF OF PROPOSITION A.1. Fix $P^* \in \mathcal{P}^*$ and let $0 \leq s \leq t \leq T$. Let $\mathcal{Q} = \{Q_1, Q_2, \dots\}$ be such that $A_t^* = \text{ess sup}_{R \in \mathcal{Q}} E(\eta^* | \mathcal{F}_t)$ P^* -a.s. Suppose that $Q_1 = P^*$. Set

$$(A.2) \quad Q = \sum_{m=1}^{\infty} \frac{1}{2^m} Q_m.$$

For all $R \in \mathcal{D}$, define $\tilde{E}_R(\eta^* | \mathcal{F}_t)$ Q -a.s. by setting it to $-\infty$ on the set $\Omega - C_R$ and to $E_R(\eta^* | \mathcal{F}_t)$ Q -a.s. on C_R , where $Q \ll R$ on C_R and $R(\Omega - C_R) = 0$. Set

$$(A.3) \quad Z^{(n)} = \sup_{1 \leq i \leq n} \tilde{E}_{Q_i}(\eta^* | \mathcal{F}_t),$$

and let C_i , $i = 1, \dots, n$ be a \mathcal{F}_t -measurable partition of Ω so that $Z^{(n)} = \tilde{E}_{Q_i}(\eta^* | \mathcal{F}_t)$ on C_i . Let U be the measure given by

$$U(C) = E^* \sum_{i=1}^n Q_i(C | \mathcal{F}_t) I_{C_i}.$$

Since $U \in \mathcal{P}^*$, it follows that

$$(A.4) \quad \begin{aligned} A_s^* &\geq E_U(\eta^* | \mathcal{F}_s) \\ &= E^*(Z^{(n)} | \mathcal{F}_s) \\ &\rightarrow E^*(A_t^* | \mathcal{F}_s) \end{aligned}$$

as $n \rightarrow \infty$ by monotone convergence, since $E^*|Z^{(1)}| \leq E^*|\eta^*| < \infty$ by assumption. \square

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DEPARTMENT OF STATISTICS
UNIVERSITY OF CHICAGO
5734 UNIVERSITY AVENUE
CHICAGO, ILLINOIS 60637
E-MAIL: mykland@galton.uchicago.edu