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Limit Theorems for Stochastic Processes

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${}^pH'' = 0$. Set also $B_t = \sum_{s\le t} |K_s|$, which clearly belongs to $\mathscr{V}^+$. Since $\Delta B \le |\Delta A|^{1/2}$ we deduce from the property $A^{1/2} \in \mathscr{A}^+_{\text{loc}}$ that $B \in \mathscr{A}^+_{\text{loc}}$, and $\sum_{s\le \cdot} |{}^pK_s| \le B^p$ (by 3.21), which also belongs to $\mathscr{A}^+_{\text{loc}}$. Thus $\sum_{s\le \cdot} |H''_s| \in \mathscr{A}^+_{\text{loc}}$ and by (b) there is $X'' \in \mathscr{M}_{\text{loc}}$ with $\Delta X'' = H''$.

Since $|H'|^2 \le 2|H|^2 + 2|H''|^2$, we get $C_t := \sum_{s\le t} |H'_s|^2 \le 2A_t + 2\sum_{s\le t} |H''_s|^2$, so $C_t < \infty$ for $t \in \mathbb{R}_+$. Moreover, since ${}^pH = 0$ we have ${}^pK = -{}^p(H 1_{\{|H|\le 1\}})$, so $|{}^pK| \le 1$, and $|H'| \le 2$ by construction: therefore $\Delta C_t \le 4$, and we deduce that $C \in \mathscr{A}^+_{\text{loc}}$. Then (a) yields a local martingale X' with $\Delta X' = H'$. Hence $X = X' + X''$ meets $\Delta X = H$. □

3. Now we turn to *Ito's formula*. In the following, $D_i f$ and $D_{ij} f$ denote the partial derivatives $\partial f/\partial x^i$ and $\partial^2 f/\partial x^i \partial x^j$.

4.57 **Theorem.** *Let $X = (X^1, \dots, X^d)$ be a d-dimensional semimartingale, and f a class C^2 function on $\mathbb{R}^d$. Then $f(X)$ is a semimartingale and we have*:

4.58
$$\begin{aligned} f(X_t) = f(X_0) &+ \sum_{i\le d} D_i f(X_-) \cdot X^i + \frac{1}{2} \sum_{i,j\le d} D_{ij} f(X_-) \cdot \langle X^{i,c}, X^{j,c}\rangle \\ &+ \sum_{s\le t} \left[f(X_s) - f(X_{s-}) - \sum_{i\le d} D_i f(X_{s-}) \Delta X^i_s \right] \end{aligned}$$

Of course, this formula implicitely means that all terms are well-defined. In particular the last two terms are processes with finite variation (the first one is continuous, the second one is "purely discontinuous").

Formula 4.58 is also valid when f is complex-valued: take the real and purely imaginary parts separately.

Proof. To simplify notation somewhat, with any C^2 function f on $\mathbb{R}^d$ we associate the C^1 function $\hat f$ on $\mathbb{R}^d \times \mathbb{R}^d$ defined by

$$\hat f(x, y) = f(x) - f(y) - \sum_{j\le d} D_j f(y)(x^j - y^j),$$

where x^j denotes the j^{th} component of x.

(i) We first prove the result when f is a polynomial on $\mathbb{R}^d$. It suffices to consider the case of monomials and, by induction on the degree and since the result is trivially true for constant functions, it suffices to prove the following: let g be a function meeting $g(X) \in \mathscr{S}$ and 4.58, then $f(x) = x^k g(x)$ also satisfies $f(X) \in \mathscr{S}$ and 4.58.

Since $g(X) \in \mathscr{S}$ and $X^k \in \mathscr{S}$, we have $f(X) \in \mathscr{S}$ by 4.47b and 4.45, and we also have $f(X) = f(X_0) + X^k_- \cdot g(X) + g(X_-) \cdot X^k + [X^k, g(X)]$. Now g satisfies 4.58, hence (using several times 4.36 and 4.37) we obtain:

$$\begin{aligned} (1) \quad f(X) = f(X_0) &+ \sum_{i\le d} (X^k_- D_i g(X_-)) \cdot X^i + \frac{1}{2} \sum_{i,j\le d} (X^k_- D_{ij} g(X_-)) \cdot \langle X^{i,c}, X^{j,c}\rangle \\ &+ \sum_{s\le \cdot} X^k_{s-} \hat g(X_s, X_{s-}) + g(X_-) \cdot X^k + [X^k, g(X)]. \end{aligned}$$