Statistical Methods for High Frequency Data
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Excercise Set 3

Make the assumptions in Section 3.2 of the notes. Your data is $X_0, X_{t_{n,1}}, ..., X_{t_{n,i}}, ..., X_{t_{n,n}} = X_T$. For simplicity we write t_i to mean $t_{n,i}$. We shall refer to the usual grid $\mathcal{G}_n = \{0 = t_{n,0}, t_{n,1}, ..., t_{n,n} = T\}$.

1. (Simple subsampling.)

   (a) Refer to Problem 1 on HW 2. Find the asymptotic distribution of $L_t^{(n)}$ when suitably normalized. Also recall how to estimate $\widehat{AVAR}$ of the the asymptotic variance $AVAR$.

   (b) Under suitable normalization, $L_T^{(n)}/\sqrt{\widehat{AVAR}}$ is asymptotically N(0,1). Use the Apple data for December 3, to calculate this statistic.

   (c) Suppose you wish to test whether this data is generated by a semi-martingale. What is the null hypothesis, and what is the p-value with the statistic you computed in the previous question.

2. (Spot volatility.) Suppose we would like to estimate the spot volatility σ_t^2 at time $t \in (0, T)$. Define an estimator by

$$\hat{\sigma}_t^2 \;=\; \frac{1}{h}\left([X,X]_t^{\mathcal{G}_n} - [X,X]_{t-h}^{\mathcal{G}_n}\right) \tag{1}$$

   For simplicity, assume that observation points are equidistant, and that σ_t^2 itself is an Itô process. We shall suppose that $h = h_n \to 0$ as $n \to \infty$.

   (a) To analyze this estimator, show the decomposition

$$\hat{\sigma}_t^2 - \sigma_t^2 \;=\; \underbrace{\frac{1}{h}(M_t - M_{t-h})}_{\text{discretization error}} \;+\; \frac{1}{h}\underbrace{\int_{t-h}^{t}(t-h-s)d\sigma_s^2}_{\text{``bias''}} \;+\; O_p((nh)^{-1}). \tag{2}$$

   Here, M is the error process discussed in Section 2.3 of the notes.

   (b) Find the quadratic variations and covariation of the two terms in (2).

   (c) Find α and a relationship between h_n and n so that $n^\alpha(\hat{\sigma}_t^2 - \sigma_t^2)$ converges in law (stably) to a limit where both the terms in (2) are represented.

(d) Explain why is the fastest obtainable speed of convergence. (In other words, if β is such that $n^{\beta}(\hat{\sigma}_t^2 - \sigma_t^2) = O_p(1)$, then $\beta \leq \alpha$.

(e) Suppose $u \neq t$, $u \in (0, T)$, what is the joint limit of $(n^{\alpha}(\hat{\sigma}_t^2 - \sigma_t^2), n^{\alpha}(\hat{\sigma}_u^2 - \sigma_u^2))$?

(f) Propose and discuss a consistent estimator of the volatility of volatility $[\sigma^2, \sigma^2]_T$.