$$P \text{ AND } P^*$$

$$(B_t, S_t^{(t)}, \ldots, S_t^{(p)}): \text{ securities}$$

$P$: actual probability

$P^*$: risk neutral probability

Realationship: mutual absolute continuity $P \sim P^*$

For example:

$P: \ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$

$P^*: \ dS_t = \mu_t^* S_t dt + \sigma_t^* S_t dW_t^*$

Money market bond numeraire:

$$\mu_t^* = r_t$$

$$Q: \ \sigma_t^* = ?? \quad A: \ \sigma_t^{*2} = \sigma_t^2. \ Why?$$
CONTINUOUS CASE: $\sigma_t$ AND $\sigma^*_t$

$P : \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$

$P^* : \quad dS_t = \mu^*_t S_t dt + \sigma^*_t S_t dW^*_t$

$P : \quad d\log S_t = \ldots dt + \sigma_t dW_t$

$\therefore \quad (d\log S_t)^2 = \sigma^2_t (dW_t)^2 = \sigma^2_t dt$

$\quad d[\log S, \log S]_t$

$P^* : \quad$ same argument : \quad $d[\log S, \log S]_t = \sigma^*_t dt$

Process same under $P, P^*$:

$\sigma^*_t dt = d[\log S, \log S]_t = \sigma^2_t dt$

$\therefore \quad \sigma^*_t = \sigma^2_t$

In particular: if $\sigma$ is constant:

$\sigma^* = \sigma$

If numeraire = Money Market Bond:

$dS_t = r_t S_t dt + \sigma_t S_t dW^*_t$

$\tilde{d}\tilde{S}_t = \sigma_t \tilde{S}_t dW^*_t$
CONTINUOUS CASE: THE MARKET PRICE OF RISK

\[ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t \]

\[ dS_t = \mu^*_t S_t dt + \sigma_t S_t dW^*_t \]

\[
\mu_t S_t dt + \sigma_t S_t dW_t = \mu^*_t S_t dt + \sigma_t S_t dW^*_t \\
\Rightarrow: \mu_t dt + \sigma_t dW_t = \mu^*_t dt + \sigma_t dW^*_t \\
\Rightarrow: \frac{\mu_t - \mu^*_t}{\sigma_t} dt + dW_t = dW^*_t \\
\underbrace{\lambda_t}_{\lambda_t}
\]

Change from \( P \) to \( P^* \):

Market price of risk

\[
\underbrace{dW^*_t}_{P^* - BM} = \underbrace{dW_t + \lambda_t dt}_{P - BM}
\]
OTHER NUMERAIRE

\[d \log S_t = \ldots dt + \sigma_t dW_t \quad P, P^*\]
\[d \log \Lambda_t = \ldots dt + \gamma_t dV_t\]

\[\tilde{S}_t = \frac{S_t}{\Lambda_t} \Rightarrow \log \tilde{S}_t = \log S_t - \log \Lambda_t\]

\[\Rightarrow\]

\[(d \log \tilde{S}_t)^2 = (d \log S_t)^2 + (d \log \Lambda_t)^2 - 2(d \log S_t)(d \log \Lambda_t)\]
\[= \sigma_t^2 (dW_t)^2 + \gamma_t^2 (dV_t)^2 - 2\sigma_t \gamma_t (dW_t)(dV_t)\]
\[= (\sigma_t^2 + \gamma_t^2 - 2\sigma_t \gamma_t \rho_t) dt\]

\[\rho_t = \frac{1}{dt} d[W, V]_t = \text{correlation} (dW, dV)\]

(can be random)

\[\tilde{\sigma}_t^2 = \sigma_t^2 + \gamma_t^2 - 2\sigma_t \gamma_t \rho_t \quad P, P^*\]
\[\neq \sigma_t^2 \text{ ex } \gamma_t = 0 : \text{ MONEY MARKET BOND}\]
OTHER NUMERAIRE

\[ d \log S_t = \ldots dt + \sigma_t dW_t \]

\[ d \log \Lambda_t = \ldots dt + \gamma_t dV_t \]

so \[ d \log \tilde{S}_t = \ldots dt + \sigma_t + \sigma_t dW_t - \gamma_t dV_t \]

\[ = \ldots dt + \tilde{\sigma}_t d\tilde{W}_t \] previous page

or \[ d\tilde{S}_t = \ldots \tilde{S}_t dt + \tilde{\sigma}_t \tilde{S}_t d\tilde{W}_t \]

\[ = 0 \text{ if } P^* \text{ for numeraire } \Lambda \]

\[ \tilde{\sigma}_t d\tilde{W}_t = \sigma_t dW_t - \gamma_t dV_t \]

or

\[ d\tilde{W}_t = \frac{\sigma_t}{\tilde{\sigma}_t} dW_t - \frac{\gamma_t}{\tilde{\sigma}_t} dV_t \]

Yet another Brownian motion
**P** depend on numeraire

Λₜ, Bₜ: numeraires, suppose P* same

\[ M_t = \frac{\Lambda_t}{B_t} = P^* - MG \]

\[ \frac{1}{M_t} = \frac{B_t}{\Lambda_t} = P^* - MG \]

\[ d \left( \frac{1}{M_t} \right) = - \frac{1}{M_t^2} dM_t + \frac{1}{M_t^3} d[M, M]_t \]

\[ d[M, M]_t = dMG \]

\[ dt \text{ term} \]

Uniqueness of Doob-Meyer \( \Rightarrow \)

\[ [M, M]_t = 0 \]

\( \Rightarrow M_t \) constant if continuous

\( \Rightarrow \Lambda, B \) are the same
THE RADON-NIKODYM THEOREM

**Theorem.** Let $P, Q$ be probabilities, $Q ≪ P$. Then

\[ \exists \text{r.v. } \frac{dQ}{dP} : \]

if $E_Q |X| < \infty$, then

\[ E_Q X = E_P \left( X \frac{dQ}{dP} \right). \]

$\frac{dQ}{dP}$ is unique $P$ – a.s.

**Proof and elaborations:** Billingsley, Section 32.

**Uniqueness:** Suppose both $Y, Z$ satisfy conditions on $\frac{dQ}{dP}$. Set

\[ A = \{ \omega : Y(\omega) > Z(\omega) \}. \]

Then

\[ Q(A) = E_P Y I_A > E_P Z I_A = Q(A) \]

unless $P(A) = 0$.

The finite case: $\mathcal{F} = \sigma(\mathcal{P})$:

\[ \frac{dQ}{dP}(\omega) = \frac{Q(A)}{P(A)} \]

when $\omega \in A \in \mathcal{P}$
CONTINUOUS DISTRIBUTIONS

\[ P(Z \in A) = \int_A f(z)\,dz \quad Q(Z \in A) = \int_A g(z)\,dz. \]

Then:

\[ \frac{dQ}{dP}(\omega) = \frac{g(Z)}{f(Z)}. \]

Proof:

\[ E_P \frac{g(Z)}{f(Z)} I(Z \in A) = \int \frac{g(z)}{f(z)} I(z \in A) f(z)\,dz \]

\[ = \int I(z \in A) g(z)\,dz \]

\[ = Q(A). \]

Generally: \( Z_1, \ldots, Z_p \)

\[ P(A) = \int_A f(z_1, \ldots, z_p)\,dz_1 \ldots dz_p \]

\[ Q(A) = \int_A g(z_1, \ldots, z_p)\,dz_1 \ldots dz_p \]

\[ \frac{dQ}{dP}(\omega) = \frac{g(Z_1, \ldots, Z_p)}{f(Z_1, \ldots, Z_p)}. \]
NORMAL PROCESSES WITH DRIFT

\[ P_\theta : \quad X_{t+1} = X_t + \theta_t + \sigma \epsilon_{t+1} \]

\( \epsilon_{t+1} \sim N(0, 1), \quad \Pi \mathcal{F}_t. \)

\[ f_\theta(x_{t+1} | \mathcal{F}_t) = \frac{1}{\sqrt{2\pi} \sigma} \exp \left\{ -\frac{(x_{t+1} - X_t - \theta_t)^2}{2\sigma^2} \right\} \]

\[ f_\theta(x_1, \ldots, x_t) = (2\pi\sigma^2)^{-\frac{t}{2}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (x_{u+1} - x_u - \theta_u)^2 \right\} \]

\[ \frac{dP_\theta}{dP_0} = \frac{f_\theta(X_1, \ldots, X_t)}{f_0(X_1, \ldots, X_t)} \]

\[ = \exp \left\{ -\frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (X_{u+1} - X_u - \theta_u)^2 + \frac{1}{2\sigma^2} \sum_{u=0}^{t-1} (X_{u+1} - X_u)^2 \right\} \]

\[ = \exp \left\{ \frac{1}{\sigma^2} \sum_{u=0}^{t-1} \theta_u (X_{u+1} - X_u) - \frac{1}{2\sigma^2} \sum_{u=0}^{t-1} \theta_u^2 \right\} \]

\[ = \exp \left\{ Y_t - \frac{1}{2} [Y, Y]_t \right\} \]

where: \( Y_t = \frac{1}{\sigma^2} \int_0^t \theta_u dX_u \) and \( \theta_u = \theta_t \) for \( t < u \leq t + 1 \)
GIRSANOV’S THEOREM

\[
\begin{align*}
\frac{dX_t}{dt} & = dW_t + \theta_t dt \\
\frac{dW_t}{dt} & = \theta_t dt
\end{align*}
\]

\[
\begin{align*}
P - BM & = Q - BM \\
\frac{dQ}{dP} & = \exp(M_T - \frac{1}{2}[M, M]_T) \\
\frac{dP}{dQ} & = \exp(-M_T + \frac{1}{2}[M, M]_T)
\end{align*}
\]

where \( M_t = \int_0^t \theta_u dX_u = P - MG \)

\[
\begin{align*}
\frac{dP}{dQ} & = \exp(-M_T + \frac{1}{2}[M, M]_T) \\
& = \exp(- \int_0^T \theta_u [dW_u + \frac{1}{2} \theta_u du] + \frac{1}{2} \int_0^T \theta_u^2 du) \\
& = \exp(- \int_0^T \theta_u dW_u - \frac{1}{2} \int_0^T \theta_u^2 du) \\
& = \exp(\tilde{M}_T - \frac{1}{2} [\tilde{M}, \tilde{M}]_T) \text{ where } \tilde{M}_t = - \int_0^t \theta_u dW_u \\
\end{align*}
\]

\( Q - MG \) as function of \( T \)
SMALL INCREMENTS, $\theta$ CONSTANT

$P_\theta : \quad X_{t+\Delta} = X_t + \theta \Delta + \sigma (B_{t+\Delta} - B_t)$

$\theta$ replaced by $\theta \Delta \quad \sigma$ replaced by $\sigma \sqrt{\Delta}$

$$
\frac{dP_\theta}{dP_0} = \exp \left\{ \frac{\theta \Delta}{\sigma^2 \Delta} X_{N\Delta} - \frac{1}{2} \frac{\theta^2 \Delta^2}{\sigma^2 \Delta} N \right\}
$$

$$
= \exp \left\{ \frac{\theta}{\sigma^2} X_T - \frac{1}{2} \frac{\theta^2}{\sigma^2} T \right\} \quad T = \Delta N
$$

If $T$ fixed, $N \to \infty$, $\Delta \to 0$:

$P_\theta : \quad X_t = \theta t + \sigma B_t$

BROWNIAN MOTION WITH DRIFT

Form of $\frac{dP_\theta}{dP_0}$: simplest case of Girsanov’s Theorem.
APPLICATION:
EVALUATION OF THE EURO-RUSSIAN OPTION

Recall: \( V_0 = e^{-rT} E^* \text{[payoff]} \)

\[
V_0 = e^{-rT} E^* f \left( \max_{0 \leq t \leq T} S_t \right) \\
= e^{-rT} E^* \left( \exp \left( \max_{0 \leq t \leq T} \log S_t \right) \right) \\
= e^{-rT} E^* \left( \exp \left( \max_{0 \leq t \leq T} \log S_t \right) \right) \\
= e^{-rT} E^* \left( \exp \left( \log S_0 + \sigma \max_{0 \leq t \leq T} (\nu t + B^*) \right) \right) \\
\sigma \nu = r - \frac{1}{2} \sigma^2 \\
= e^{-rT} E_Q f \left( \exp(\log S_0 + \sigma \max_{0 \leq t \leq T} X_t) \right) \frac{dP^*}{dQ} \\
P^* : \quad X_t = \nu t + B^* \\
Q : \quad X_t \text{ is Brownian motion.} \\
\frac{dP^*}{dQ} = \exp\{\nu X_T - \frac{1}{2} \nu^2 T\}. \\

Problem reduced to one involving maxima of Brownian motion.
\[ M_T = \max_{0 \leq t \leq T} X_t \]

\[ V_0 = e^{-rT} E_Q f (S_0 \exp(\sigma M_T)) \exp \left( \nu X_T - \frac{1}{2} \nu^2 T \right) \]

\[ f_{X,M}(a, b) = \frac{2(2b - a)}{\sqrt{2\pi T^3}} \exp \left\{ -\frac{(2b - a)^2}{2T} \right\} \]

\[ b \geq a, 0 \]

\[ V_0 = e^{-rT} \int \int f(S_0 \exp(\sigma b)) \exp \left( \nu a - \frac{1}{2} \nu^2 T \right) f_{X,M}(a, b) \, da \, db \]
ADDITIVE AND MULTIPLICATIVE MARTINGALES

$M_t$ IS A MARTINGALE

$$Z_t = \exp(M_t - \frac{1}{2}[M, M]_t) = f(X_t)$$

Itô:

$$dZ_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d[X, X]_t$$

$$= \exp(X_t)\left[dX_t + \frac{1}{2}d[X, X]_t\right]$$

$$= Z_t dM_t = MG$$

Since

$$dX_t + \frac{1}{2}d[X, X]_t$$

$$= dM_t - \frac{1}{2}d[M, M]_t + \frac{1}{2}d[M, M]_t = dM_t$$

In particular:

$$EZ_T = EZ_0 = E1 = 1$$
THE DISTRIBUTION OF HITTING TIMES

\[ P^* : \quad X_t = \nu t + \sigma W_t \]  and \[ Q : \quad X_t = \sigma W_t^* \]

\[ \frac{dP^*}{dQ} \bigg|_t = \exp \left\{ \frac{\nu}{\sigma^2} X_t - \frac{1}{2} \frac{\nu^2}{\sigma^2} t \right\}. \]

Martingale property:

\[ \frac{dP^*}{dQ} \bigg|_{\tau} = \exp \left\{ \frac{\nu}{\sigma^2} X_{\tau} - \frac{1}{2} \frac{\nu^2}{\sigma^2} \tau \right\}. \]

A special case

\[ \tau = \inf \{ t : \quad X_t = b \}. \]

\[ P^*(\tau \leq u) = E^* I_{\{\tau \leq u\}} \]

\[ = E_Q I_{\{\tau \leq u\}} \exp \left\{ \frac{\nu}{\sigma^2} X_{\tau \wedge u} - \frac{1}{2} \frac{\nu^2}{\sigma^2} (\tau \wedge u) \right\} \]

\[ = E_Q I_{\{\tau \leq u\}} \exp \left\{ \frac{\nu}{\sigma^2} b - \frac{1}{2} \frac{\nu^2}{\sigma^2} \tau \right\} \]

\[ = \int_0^u \exp \left( \frac{\nu}{\sigma^2} b - \frac{1}{2} \frac{\nu^2}{\sigma^2} t \right) f_{Q, \tau}(t) \, dt \]
\[ P^*(\tau \leq u) = \int_0^u \exp \left( \frac{\nu}{\sigma^2} b - \frac{1}{2} \frac{\nu^2}{\sigma^2} t \right) f_{Q,\tau}(t) \, dt \]

or:

\[ f_{P^*,\tau}(t) = \exp \left( \frac{\nu}{\sigma^2} b - \frac{1}{2} \frac{\nu^2}{\sigma^2} t \right) f_{Q,\tau}(t). \]

From distribution of maximum:

\[ f_{Q,\tau}(t) = \frac{|b|}{\sqrt{2\pi\sigma^2 t^3}} \exp \left\{ -\frac{b^2}{2\sigma^2 t} \right\}, \quad t \geq 0 \]

**EXAMPLE: DOWN AND IN OPTIONS**

\[ \eta = \begin{cases} (S_T - K)^+ & \text{of } \min_{0 \leq t \leq T} S_t \leq K' \\ 0 & \text{otherwise} \end{cases} \]

\[ X_t = \log S_t - \log S_0 \quad b = \log K' - \log S_0 \]

price = \( E^* e^{-rT} \eta \)

\[ = E^* E^* \left[ e^{-rT} \eta \mid \mathcal{F}_\tau \right] I(\tau \leq T) \]

\[ = E^* e^{-rT} E^* \left[ e^{-r(T-\tau)} \eta \mid \mathcal{F}_\tau \right] I(\tau \leq T) \]

\[ = E^* e^{-rT} BS(T - \tau) I(\tau \leq T) \]

\[ = \int_0^T e^{-rt} BS(T - t) f_{P^*,\tau}(t) \, dt \]