MARTINGALES: VARIANCE

SIMPLEST CASE: $X_t = Y_1 + \ldots + Y_t$ IID SUM

$EY = 0 \quad \text{Var} (Y) = \sigma^2$

$V_t = X_t^2 - \sigma^2 t$ IS A MG:

$V_{t+1} = (X_t + Y_{t+1})^2 - \sigma^2(t + 1)$

$= V_t + 2X_t Y_{t+1} + Y_{t+1}^2 - \sigma^2$

$E(V_{t+1} \mid \mathcal{F}_t) = V_t + 2X_t E(Y_{t+1} \mid \mathcal{F}_t)$

$= 0$

$+ E(Y_{t+1}^2 \mid \mathcal{F}_t) - \sigma^2$

$= 0$

$= V_t$
QUADRATIC VARIATION (Q.V.)

\[ X_t = Y_1 + \ldots + Y_t : MG \text{ (not iid)} \]

**OBSERVED Q.V.:** \[ [X, X]_t = Y_1^2 + \ldots + Y_t^2 \]

- \( V_t = X_t^2 - [X, X]_t : MG \)

\[
V_{t+1} = (X_t + Y_{t+1})^2 - [X, X]_{t+1}
= X_t^2 + 2X_tY_{t+1} + Y_{t+1}^2 - ([X, X]_t + Y_{t+1}^2)
= V_t + 2X_tY_{t+1}
\]

\[
E(V_{t+1} \mid \mathcal{F}_t) = V_t + 2X_tE(Y_{t+1} \mid \mathcal{F}_t) = V_t
\]

**PREDICTABLE Q. V.:**

\[
\langle X, X \rangle_t = \text{Var} (Y_1 \mid \mathcal{F}_0) + \ldots + \text{Var} (Y_t \mid \mathcal{F}_{t-1})
\]

- \([X, X]_t - \langle X, X \rangle_t = \sum_{i=1}^{t} Y_i^2 - E(Y_i^2 \mid \mathcal{F}_{i-1}) = MG\]
- \(X_t^2 - \langle X, X \rangle_t = MG, \quad \text{TOO}\)
CONTINUOUS MARTINGALES

* TWO CONTINUITIES:

— TIME ITSELF:

\[ M_t, \quad 0 \leq t \leq T \quad (\text{or } 0 \leq t < \infty) \]

— PROCESS PATH:

\[ t \to M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION OF TIME} \]

* QUADRATIC VARIATION FOR CONTINUOUS MGs:

\[ \Delta M_{t_i} = M_{t_{i+1}} - M_{t_i} \]

\[
[M, M]_t = \lim_{\max \Delta t_i \to 0} \sum_{t_{i+1} \leq t} \Delta M_{t_i}^2 \]

SAME AS

\[
\langle M, M \rangle_t = \lim_{\max \Delta t_i \to 0} \sum_{t_{i+1} \leq t} \text{Var} \left( \Delta M_{t_i} \mid \mathcal{F}_{t_i} \right) \]
“PROOF” THAT $\langle M, M \rangle_t = [M, M]_t$:

**BEFORE CONVERGENCE:**

$$L_t = \sum_{t_{i+1} \leq t} \Delta M_{t_i}^2 - \sum_{t_{i+1} \leq t} \text{Var} (\Delta M_{t_i} \mid \mathcal{F}_{t_i})$$

$L$ IS A MARTINGALE, AND

$$\Delta L_{t_i} = \Delta M_{t_i}^2 - \text{Var} (\Delta M_{t_i} \mid \mathcal{F}_{t_i})$$

SO

$$E[\Delta L_{t_i}^2] = E[(\Delta M_{t_i}^2 - \text{Var} (\Delta M_{t_i} \mid \mathcal{F}_{t_i}))^2]$$

$$\leq 2E[(\Delta M_{t_i}^4 + \text{Var} (\Delta M_{t_i} \mid \mathcal{F}_{t_i})^2]$$

$$= 2E[\Delta M_{t_i}^4 + E(\Delta M_{t_i}^2 \mid \mathcal{F}_{t_i})]^2]$$

$$\leq 2E[\Delta M_{t_i}^4 + E(\Delta M_{t_i}^4 \mid \mathcal{F}_{t_i})]$$

$$= 4E[\Delta M_{t_i}^4]$$

SO

$$\text{Var} (L_{t_i}) = E[L, L]_{t_i}$$

$$\leq \sum_{t_{i+1} \leq t} 4E[\Delta M_{t_i}^4]$$

$$\leq 4E \left[ \max_j \Delta M_{t_j}^2 \sum_{t_{i+1} \leq t} \Delta M_{t_i}^2 \right]$$

$$= 4E \left[ \max \Delta M_{t_i}^2 [M, M]_t \right]$$

$$\rightarrow 0 \text{ as } \max \Delta t_i \rightarrow 0 \text{ BY CONTINUITY}$$

QED (CF. SHREVE II, THM 3.4.3 (p. 102))
CONTINUOUS MG HAVE INFINITE PATH

\[ TV(M)_t = \lim_{t_{i+1} < t} \sum_{t_i < t} |\Delta M_{t_i}| \]

QUADRATIC VARIATION:

\[ \sum_{t_{i+1} < t} \Delta M_{t_i}^2 \leq \max_{t_{i+1} < t} |\Delta M_{t_1}| \]

\[ \sum_{t_{i+1} < t} |\Delta M_{t_i}| \]

\( [M,M]_t > 0 \)

\( 0 \)

\( TV(M)_t \)

(CONTINUITY)

HENCE TV(M)_t IS INFINITE
THE THEORY IS ONLY APPROXIMATELY CORRECT

Plot of realized volatility (quadratic variation of log price) for Alcoa Aluminum for January 4, 2001. The data is from the TAQ database. There are 2011 transactions on that day, on average one every 13.365 seconds. The most frequently sampled volatility uses all the data, and this is denoted as “frequency= 1”. “Frequency=2” corresponds to taking every second sampling point. Because this gives rise to two estimators of volatility, we have averaged the two. And so on for “frequency= k” up to 20. The plot corresponds to the average realized volatility. Volatilities are given on an annualized and square root scale.
SOLUTION TO PROBLEM

\[ \log S_{t_i} = \text{martingale} + \text{constant} \times t + \epsilon_i \]

\( t_i = \) transaction time \( \# i \)
\( S_{t_i} = \) value of stock at time \( t_i \)
\( \epsilon_i: \) (almost) independent noise (very small)

STOCK BEHAVES LIKE A MEASUREMENT OF UNDERLYING SEMI-MARTINGALE
\( \epsilon_i \) is too small for effective arbitrage
(ADDITIVE) BROWNIAN MOTION $W_t$

ORIGINAL DEFINITION:
(1) $W_0 = 0$
(2) $t \to W_t$ IS CONTINUOUS
(3) HAS INDEPENDENT INCREMENTS
(4) $W_{t+s} - W_s \sim N(0, t)$

ALTERNATIVE DEFINITION (“LEVY’S THEOREM”):
(i) $W_0 = 0$
(ii) $W_t$ is MG, CONTINUOUS
(iii) $[W, W]_t = t$

PROOF THAT ORIG DEF $\Rightarrow$ (iii):

BEFORE CONVERGENCE:

$$
\Delta \langle W, W \rangle_{t_i} = \langle W, W \rangle_{t_{i+1}} - \langle W, W \rangle_{t_i}
= \text{Var} (W_{t_{i+1}} - W_{t_i} \mid \mathcal{F}_{t_i})
= t_{i+1} - t_i
$$

AND SO $\langle W, W \rangle_t \approx t$ AS $\Delta t_i \to 0$

FROM p. 3-4: $[W, W]_t = \langle W, W \rangle_t$ QED
MARKOV PROPERTY

FOR $t \geq s$, and $W_s = w$

$$E(f(W_t) \mid \mathcal{F}_s) = E(f(W_t - W_s + W_s) \mid \mathcal{F}_s)$$
$$= E(f(W_t - W_s + w) \mid \mathcal{F}_s)$$
$$= E(f(W_t - W_s + w))$$

since $W_t - W_s$ independent of $\mathcal{F}_s$

$$= E(f(\sqrt{t-s}Z + w))$$

where $Z$ is $N(0, 1)$

$$= g(w) = g(W_s)$$

HERE

$$g(w) = \int_{-\infty}^{\infty} f(\sqrt{t-s}z + w)\phi(z)dz$$

AS ALWAYS: WE CAN IN PARTICULAR TAKE

$$f(W) = I_{\{W \in A\}}$$

TO GET (FOR SOME $g$):

$$P(W_t \in A \mid \mathcal{F}_s) = g(W_s)$$

CONDITIONAL DISTRIBUTION OF $W_t$ GIVEN $\mathcal{F}_s$ DEPENDS ONLY ON $W_s$
FIRST PASSAGE TIMES

$m$ IS A BARRIER, $m > 0$

$$\tau = \inf\{t \geq 0 : W_t = m\}$$

$= \text{FIRST TIME THAT } W_t \text{ HITS } m$

$\tau$ IS A STOPPING TIME

B-S MARTINGALE: $\tilde{S}_t = \exp(\sigma W_t - \frac{1}{2} \sigma^2 t)$

- $W_{t \wedge \tau} \leq m$: ON THE SET $\{\tau = \infty\}$:

$$\tilde{S}_{t \wedge \tau} \leq \exp(\sigma m - \frac{1}{2} \sigma^2 (t \wedge \tau))$$

$\rightarrow 0$ when $t \rightarrow \infty$

- OPTIONAL STOPPING:

$$1 = E[\tilde{S}_{t \wedge \tau}]$$

$$\rightarrow E[\tilde{S}_\tau \mathbb{1}_{\{\tau < \infty\}}] \text{ when } t \rightarrow \infty$$

BECAUSE OF DOMINATED CONVERGENCE:

$W_{t \wedge \tau} \leq m \AND t \wedge \tau \geq 0 \Rightarrow \tilde{S}_{t \wedge \tau} \leq \exp(\sigma m)$
FIRST PASSAGE TIMES (cont’d)

\[ 1 = E[\tilde{S}_\tau I_{\{\tau < \infty\}}] \]
\[ = E[\exp(\sigma W_\tau - \frac{1}{2} \sigma^2 \tau)I_{\{\tau < \infty\}}] \]
\[ = E[\exp(\sigma m - \frac{1}{2} \sigma^2 \tau)I_{\{\tau < \infty\}}] \]

LET \( \sigma \to 0: 1 = EI_{\{\tau < \infty\}} = P(\tau < \infty) \)

SIMPLIFIED FORMULA:

\[ 1 = E[\exp(\sigma m - \frac{1}{2} \sigma^2 \tau)] = \exp(\sigma m)E[\exp(-\frac{1}{2} \sigma^2 \tau)] \]

OR:

\[ E[\exp(-\frac{1}{2} \sigma^2 \tau)] = \exp(-\sigma m) \]

SET \( \alpha = \sigma^2 / 2 \)

\[ E[\exp(-\alpha \tau)] = \exp(-|m|\sqrt{2\alpha}) \]

THIS IS THE “LAPLACE TRANSFORM” OF THE DISTRIBUTION OF \( \tau \)
LAPLACE TRANSFORM:

\[ E[\exp(-\alpha \tau)] = \exp(-|m|\sqrt{2\alpha}) \]

COROLLARY: \( E[\tau] = \infty \)

BECAUSE:

\[
\frac{d}{d\alpha} E[\exp(-\alpha \tau)] = -E[\tau \exp(-\alpha \tau)] \to -E[\tau] \text{ as } \alpha \to 0 \\
\frac{d}{d\alpha} E[\exp(-\alpha \tau)] = -\frac{|m|}{\sqrt{2\alpha}} \exp(-|m|\sqrt{2\alpha}) \to -\infty \text{ as } \alpha \to 0
\]

QED
REFLECTION PRINCIPLE

AS BEFORE:

\[ \tau = \inf\{t \geq 0 : W_t = m\} \]

= FIRST TIME THAT \( W_t \) HITS \( m \)

PRINCIPLE (for \( 0, w \leq m \)):

\[ P(\tau \leq t \text{ and } W_t \leq w) = P(\tau \geq 2m - w) \]

REASON FOR THIS FORMULA

- STRONG MARKOV PROPERTY:

\( W_t^* = W_{t+\tau} - W_\tau, \ t \geq 0, \) IS A BROWNIAN MOTION

(PROOF: OPTIONAL STOPPING THEOREM; ALTERNATIVE DEFINITION OF BROWNIAN MOTION)

- THEN BY SYMMETRY, ON \( \{\tau \leq t\} \), FOR \( x \geq 0 \):

\[
P(W_t - m \leq -x \mid \mathcal{F}_\tau) = P(W_t - W_\tau \leq -x \mid \mathcal{F}_\tau) = P(W_t - W_\tau \geq x \mid \mathcal{F}_\tau) = P(W_t - m \geq x \mid \mathcal{F}_\tau)
\]
• **Rewrite as Indicator Function:**

\[
E(I_{\{W_t - m \leq -x\}} \mid \mathcal{F}_\tau) = E(I_{\{W_t - m \geq x\}} \mid \mathcal{F}_\tau)
\]

• **And so**

\[
E(I_{\{W_t - m \leq -x\}}I_{\{\tau \leq t\}}) = E[E(I_{\{W_t - m \leq -x\}}I_{\{\tau \leq t\}}) \mid \mathcal{F}_\tau)]
\]

(tower property)

\[
= E[E(I_{\{W_t - m \leq -x\}} \mid \mathcal{F}_\tau)]I_{\{\tau \leq t\}}
\]

\[
= E[E(I_{\{W_t - m \geq x\}} \mid \mathcal{F}_\tau)]I_{\{\tau \leq t\}}
\]

\[
= E[E(I_{\{W_t - m \geq x\}}I_{\{\tau \leq t\}}) \mid \mathcal{F}_\tau]
\]

\[
= E[E(I_{\{W_t - m \geq x\}} \mid \mathcal{F}_\tau)]
\]

since \(\{W_t - m \geq x\} \subseteq \{\tau \leq t\}\)

\[
= E(I_{\{W_t - m \geq x\}})
\]

• **Rewrite as Probabilities:**

\[
P(W_t - m \leq -x \text{ AND } \tau \leq t) = P(W_t - m \geq x)
\]

• **Substitute** \(w = m - x\) **to obtain principle** (QED)
CONSEQUENCE: DISTRIBUTION OF $\tau$

$$P(\tau \leq t) = \frac{2}{\sqrt{2\pi}} \int_{\frac{|m|}{\sqrt{t}}}^{\infty} \exp\left\{-\frac{y^2}{2}\right\} dy \text{ for } t \geq 0 \quad (*)$$

BECAUSE: SET $w = m$ IN REFLECTION PRINCIPLE, OBTAIN:

$$P(\tau \leq t \text{ AND } W_t \leq m) = P(W_t \geq m)$$

ALSO, SINCE $\{W_t \geq m\} \subseteq \{\tau \leq t\}$:

$$P(\tau \leq t \text{ AND } W_t \geq m) = P(W_t \geq m)$$

ADD THESE TWO:

$$P(\tau \leq t) = P(\tau \leq t \text{ AND } W_t \leq m) + P(\tau \leq t \text{ AND } W_t \geq m)$$

$$= P(W_t \geq m) + P(W_t \geq m)$$

$$= 2P(W_t \geq m)$$

THIS SHOWS (*)

DENSITY: $f_\tau(t) = \frac{d}{dt} P(\tau \leq t) = \frac{|m|}{t\sqrt{2\pi t}} \exp\left\{-\frac{m^2}{2t}\right\}$
DISTRIBUTION OF
MAXIMUM OF BROWNIAN MOTION

\[ M_t = \max_{0 \leq s \leq t} W_s \]

USE

\[ M_t \geq m \text{ IF AND ONLY IF } \tau \leq t \]

FROM PREVIOUS PAGE:

\[ P(M_t \geq m) = P(\tau \leq t) = 2P(W_t \geq m) = P(|W_t| \geq m) \]

THEREFORE:

\[ M_t \text{ HAS SAME DISTRIBUTION AS } |W_t| \]
JOINT DISTRIBUTION
BROWNIAN MOTION AND ITS MAXIMUM

\[ M_t = \max_{0 \leq s \leq t} W_s \]

USE

\[ M_t \geq m \text{ IF AND ONLY IF } \tau \leq t \]

REFLECTION PRINCIPLE BECOMES \((0, w \leq m)\):

\[
P(M_t \geq m \text{ and } W_t \leq w) = P(\tau \leq t \text{ and } W_t \leq w) = P(W_t \geq 2m - w)
\]

REWRITE WITH DENSITIES

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{w} f_{M(t),W(t)}(x,y)\,dx\,dy = \frac{1}{\sqrt{2\pi t}} \int_{2m-w}^{\infty} \exp\left\{-\frac{z^2}{2t}\right\}dz
\]

\[ \frac{\partial}{\partial m}:
\]

\[
-\int_{-\infty}^{w} f_{M(t),W(t)}(m,y) = \frac{2}{\sqrt{2\pi t}} \exp\left\{-\frac{(2m-w)^2}{2t}\right\}
\]

\[ \frac{\partial}{\partial w}:
\]

\[
-f_{M(t),W(t)}(m,w) = -\frac{2(2m-w)}{t\sqrt{2\pi t}} \exp\left\{-\frac{(2m-w)^2}{2t}\right\}
\]
FINAL FORM OF DENSITY

\[ f_{M(t), W(t)}(m, w) = \frac{2(2m - w)}{t \sqrt{2\pi t}} \exp\left\{ -\frac{(2m - w)^2}{2t} \right\} \]

for 0, w ≤ m (otherwise \( f_{M(t), W(t)}(m, w) = 0 \))

CONDITIONAL DISTRIBUTION

\[ f_{M(t)|W(t)}(m \mid w) = \frac{f_{M(t), W(t)}(m, w)}{f_{W(t)}(w)} = \frac{2(2m - w)}{t \sqrt{2\pi t}} \exp\left\{ -\frac{(2m - w)^2}{2t} \right\} \]

\[ = \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{w^2}{2t} \right\} \]

\[ = \frac{2(2m - w)}{t} \exp\left\{ -\frac{2m(m - w)}{2t} \right\} \]
STOPPED $\sigma$-FIELDS

WHAT MEANING CAN WE GIVE TO “$\mathcal{F}_\tau$”? 

- $(\mathcal{F}_t)$ IS FILTRATION
- $\tau$ IS $(\mathcal{F}_t)$-STOPPING TIME
- DEFINE:

$$\mathcal{F}_\tau = \{ A \in \mathcal{F}_\infty : A \cap \{ \tau \leq t \} \in \mathcal{F}_t \text{ for all } t \}$$

$\mathcal{F}_\tau$: THE SETS $A$ WHICH CAN BE DETERMINED ONCE $\tau$ HAS OCCURRED, OR THE INFORMATION AVAILABLE AT TIME $\tau$

- $\mathcal{F}_\tau$ IS A $\sigma$-FIELD
- EXTENDED OPTIONAL STOPPING THEOREM: SUPPOSE $\tau_1$ and $\tau_2$ ARE TWO STOPPING TIMES, $\tau_1, \tau_2 \geq 0$, $\tau_1 \leq T$. SUPPOSE $(M_t)$ IS A MARTINGALE. THEN

$$E(M_{\tau_1} \mid \mathcal{F}_{\tau_2}) = M_{\tau_1 \wedge \tau_2}$$