APPROXIMATE NORMALITY

BINOMIAL MODEL: \[ S_{n+1} = \begin{cases} uS_n \\ dS_n \end{cases} \]

LOG SCALE (IID ADDITIVE INCREMENTS):
\[ \log S_n = \log S_0 + X_1 + \ldots + X_n \]
WITH \( X_i = \log(u) \) or \( = \log(d) \)

TWO TIME SCALES

CLOCK TIME: \( T \) – TIME PERIODS: \( n \)
\[ t_0 = 0 \quad t_1 = \frac{T}{n} \quad t_2 = \frac{2T}{n} \quad t_3 = \frac{3T}{n} \quad \ldots \quad t_k = \frac{kT}{n} \]
\( T \) IS FIXED – \( n \) IS A MATTER OF CHOICE

RETURN ON RISK FREE ASSET
(in clock time) \( e^{rT} = e^{\rho n} \) (in time periods)
in other words: \( \rho = r \frac{T}{n} \) \hspace{1cm} (1)

\( r \) IS FIXED – \( \rho \) DEPENDS ON \( n \)

RISK NEUTRAL MEASURE PER STEP:
\[ \pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{u - e^{r\frac{T}{n}}}{u - d} \quad \text{and} \quad \pi_n(H) = \frac{e^\rho - d}{u - d} \] \hspace{1cm} (2)
BEHAVIOR OF ADDITIVE INCREMENTS

MEAN:

\[ E(X) = \log(u)\pi(H) + \log(d)\pi(T) \]

TOTAL MEAN:

\[
E(\log(S_n) - \log(S_0)) = E(X_1) + ... + E(X_n) \\
= nE(X) \\
= n(\log(u)\pi(H) + \log(d)\pi(T))
\]

VARIANCE: \( X = \log d + (\log u - \log d)I_{\{H\}} \), and so

\[
\text{Var}(X) = (\log u - \log d)^2 \text{Var}(I_{\{H\}}) \\
= (\log u - \log d)^2 \pi(H)\pi(T)
\]

TOTAL VARIANCE:

\[
\text{Var}(\log(S_n)) = \text{Var}(X_1) + ... + \text{Var}(X_n) \\
= n \text{Var}(X_1) \\
= n(\log u - \log d)^2 \pi(H)\pi(T)
\]
WE WISH TO KEEP TOTAL MEAN, VARIANCE CONSTANT IN CLOCK TIME

\[ \nu T = E(\log S_n) \]
\[ = n(\log(u)\pi(H) + \log(d)\pi(T)) \quad (3) \]
\[ \sigma^2 T = \text{Var}(\log(S_n)) \]
\[ = n(\log u - \log d)^2\pi(H)\pi(T) \quad (4) \]

\( \sigma \) OR \( \sigma^2 \) IS VOLATILITY IN CLOCK TIME
NEED TO USE: \( \nu \approx r - \frac{1}{2}\sigma^2 \)

EQUATIONS (1)-(4) DEFINE A BINOMIAL TREE
(\( \rho, u, d, \pi(H), \pi(T) \)) ON THE BASIS OF:

\bullet VOLATILITY PER UNIT CLOCK TIME: \( \sigma^2 \)
\bullet INTEREST PER UNIT CLOCK TIME: \( r \)
\bullet \# OF UNITS OF CLOCK TIME: \( T \)
\bullet \# OF TIME PERIODS IN COMPUTATION: \( n \)
AN APPROXIMATION FOR THE CASE $r = \rho = 0$
(THE DISCOUNTED PROCESS)

UP AND DOWN STEPS:

$$u = 1 + \sqrt{\frac{\sigma^2 T}{n}} \quad \text{AND} \quad d = 1 - \sqrt{\frac{\sigma^2 T}{n}}$$

RISK NEUTRAL PROBABILITIES:

$$\pi_n(T) = \frac{u - e^\rho}{u - d} = \frac{1}{2} \quad \text{AND} \quad \pi_n(H) = \frac{e^\rho - d}{u - d} = \frac{1}{2}$$

WE SHOW THAT EQUATIONS (3)-(4) ARE APPROXIMATELY SATISFIED

WILL USE THIS APPROXIMATE BINOMIAL TREE
APPROXIMATION TO CONDITION (4):

\[ \log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots \]

\[ x = \sqrt{\frac{\sigma^2 T}{n}} : \log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2n}\sigma^2 T + \frac{1}{n\sqrt{n}} \times \ldots \]

\[ x = -\sqrt{\frac{\sigma^2 T}{n}} : \log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2n}\sigma^2 T + \frac{1}{n\sqrt{n}} \times \ldots \]

AND SO:

\[
\text{Var} \left( \log(S_n) \right) = n \left( \log u - \log d \right)^2 \pi(H) \pi(T) \\
= n \frac{1}{4} \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2n}\sigma^2 T + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \\
- \left( -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2n}\sigma^2 T + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \\
= n \frac{1}{4} \left( 2\sqrt{\frac{\sigma^2 T}{n}} + \frac{1}{n\sqrt{n}} \times \ldots \right)^2 \\
= \sigma^2 T + \frac{1}{n} \times \ldots \]
ABOUT EQUATION (3):

\[
\log(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots
\]

\[
\log(u) = \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n \sqrt{n}} \times \ldots
\]

\[
\log(d) = -\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n \sqrt{n}} \times \ldots
\]

AND SO:

\[
\nu T = E(\log(S_n) - \log(S_0))
\]

\[
= n(\log(u)\pi(H) - \log(d)\pi(T))
\]

\[
= \frac{1}{2} n \left( \sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n \sqrt{n}} \times \ldots \\
+ (-\sqrt{\frac{\sigma^2 T}{n}} - \frac{1}{2} \frac{\sigma^2 T}{n} + \frac{1}{n \sqrt{n}} \times \ldots) \right)
\]

\[
= -\frac{1}{2} \sigma^2 T + \frac{1}{\sqrt{n}} \times \ldots
\]

AS PREDICTED
HOW MUCH DO OUR RESULTS DEPEND ON $n$?

TRYING THE MATTER OUT IN R

M <- 1000  # number of simulation steps
sigma <- .2  # clock time volatility
T <- 1  # clock time duration
S0 <- 100  # initial value
piH <- 1/2  # risk neutral probability
n <- 10  # steps
u <- 1 + sqrt(T*sigma^2/n)  # up step
d <- 1 - sqrt(T*sigma^2/n)  # down step
H<- rbinom(M,n,piH)  # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1))  # check this command out!
hist(logS,freq=F)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
THE DISTRIBUTION OF $\log S_T$ STABILIZES
THE CENTRAL LIMIT PHENOMENON

**THEOREM:** SUPPOSE THAT

- $X_i, i = 1, \ldots, n$ ARE IID $P_n$
  (DISTRIBUTION CAN DEPEND ON $n$)
- $n \operatorname{Var}_n(X) \to \gamma^2$ AS $n \to \infty$

THEN

$$\sum_{i=1}^{n} X_i - nE_n(X) \xrightarrow{L} N(0, \gamma^2)$$

IN WORDS:

$\sum_{i=1}^{n} X_i - nE_n(X)$ CONVERGES IN LAW TO $N(0, \gamma^2)$

THAT IS TO SAY:

THE DISTRIBUTION OF $\sum_{i=1}^{n} X_i - nE_n(X)$ IS APPROXIMATELY NORMAL $N(0, \gamma^2)$

DENSITY OF THE NORMAL DISTRIBUTION $N(\mu, \gamma^2)$

$$\frac{d}{dx} P(N(\mu, \gamma^2) \leq x) = \frac{1}{\gamma} \phi \left( \frac{x - \mu}{\gamma} \right)$$

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} x^2\}$$
IN OUR CASE

\[ \log(S_T) - \log(S_0) = \sum_{i=1}^{n} X_i \]

\[ \gamma^2 = \sigma^2 T \]

\[ E(\log(S_T) - \log(S_0)) = nE_n(X) \approx -\frac{1}{2}\sigma^2 T \]

SO THAT

\[ \log(S_T) - \left( \log(S_0) - \frac{1}{2}\sigma^2 T \right) \]

IS APPROXIMATELY NORMAL \( N(0, \sigma^2 T) \)

OR: \( \log(S_T) \) IS APPROXIMATELY NORMAL

\( N(\log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T) \)

Note: \( Z \sim N(\mu, \gamma^2) \Leftrightarrow Z \sim \mu \sim N(0, \gamma^2) \)

\[ \Leftrightarrow \frac{Z - \mu}{\gamma} \sim N(0, 1) \]
SUPERIMPOSING THE NORMAL CURVE ON THE HISTOGRAM

```r
n <- 10       # steps
u <- 1 + sqrt(T*sigma^2/n) # up step
d <- 1 - sqrt(T*sigma^2/n) # down step
H<- rbinom(M,n,piH)       # simulation
logS <- log(S0) + log(u)*H + log(d)*(n-H)
par(mfrow=c(2,1)) # check this command out!
hist(logS,freq=F)
# compare to normal distribution
xpoints<-c(-30:30)/10
mu<-log(S0)-(sigma^2*T)/2
gamma<-sqrt(sigma^2*T)
xpoints<-c(-30:30)/10
xpoints<-mu+sigma*xpoints
density<-dnorm(xpoints,mean=mu,sd=gamma)
lines(xpoints,density)
# try again with a larger number of steps
n <- 1000
# define u, d, H, logS as above, with new n
hist(logS,freq=F)
# mu, gamma, xpoints stay the same
lines(xpoints,density)
```
NORMAL CURVE SUPERIMPOSED ON HISTOGRAMS
THE CLASSICAL CENTRAL LIMIT THEOREM

(A digression. Just so you know.)

SETUP:

$Y_1, ..., Y_n$ ARE IID, $E(Y) = 0$ AND $\text{Var}(Y) = \gamma^2$

THEN:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i \xrightarrow{L} N(0, \gamma^2)$$

PROOF:

TAKE $X_i = \frac{1}{\sqrt{n}} Y_i$

IN EARLIER THEOREM

RESULT Follows
BEHAVIOR OF OPTIONS PRICES

STEP 1: CONTINUOUS FUNCTIONS

THEOREM: IF

- $Z_n \xrightarrow{\mathcal{L}} Z$ as $n \to \infty$
- $x \to h(x)$ is a continuous function

THEN $h(Z_n) \xrightarrow{\mathcal{L}} h(Z)$ as $n \to \infty$

EXAMPLE

$Z_n = \log(S_T^{(n)}) \xrightarrow{\mathcal{L}} Z = N \left( \log(S_0) - \frac{1}{2}\sigma^2 T, \sigma^2 T \right)$

CONTINUOUS FUNCTION #1: $h(x) = e^x$:

$S_T^{(n)} = \exp\{Z_n\} \xrightarrow{\mathcal{L}} S_T^{(\infty)} = \exp\{Z\}$

CONTINUOUS FUNCTION #2: $h(x) = (x - e^{-rT}K)^+$:

$V_T^{(n)} = (S_T^{(n)} - e^{-rT}K)^+ \xrightarrow{\mathcal{L}} (S_T^{(\infty)} - e^{-rT}K)^+$

CHECK THIS IN R!
BEHAVIOR OF OPTIONS PRICES

STEP 2: THE DOMINATED CONVERGENCE THEOREM

SETUP:

- \((T_n, U_n) \xrightarrow{\mathcal{L}} (T, U)\) as \(n \to \infty\)
- \(|T_n| \leq U_n\) a.s., FOR ALL \(n\)
- \(E(U_n) \to E(U)\) as \(n \to \infty\)

THEOREM:

UNDER THESE CONDITIONS:

\[ E(T_n) \to E(T) \text{ as } n \to \infty \]

- CHECK THAT THEOREM IN SHREVE IS SPECIAL CASE
- GENERAL THEOREM:
  - See Billingsley: *Probability and Measure*
  - Deduce using Skorokhod embedding
  - For final: need only to be able to use above Theorem
BEHAVIOR OF OPTIONS PRICES

STEP 3: COMBINE THEOREMS

TAKE: \( T_n = (S_T^{(n)} - e^{-rT}K)^+ \) AND \( U_n = S_T^{(n)} \)

WE KNOW:

- \( (T_n, U_n) \xrightarrow{\mathcal{L}} (T, U) \) AS \( n \to \infty \)
- \( |T_n| \leq U_n \) a.s., FOR ALL \( n \): \((S - e^{-rT}K)^+ \leq S\)

WE NEED TO ESTABLISH

\[ E(U_n) \to E(U) \text{ AS } n \to \infty \] \hspace{1cm} (5)

IF THIS IS THE CASE, WE CAN CONCLUDE THAT

\[ \text{n step options price} = E(S_T^{(n)} - e^{-rT}K)^+ \]
\[ \to E(S_T^{(\infty)} - e^{-rT}K)^+ \] \hspace{1cm} (6)

WHERE

\[ S_T^{(\infty)} = \exp\{Z\} \]

AND

\[ Z = N \left( \log(S_0) - \frac{1}{2} \sigma^2 T, \sigma^2 T \right) \]
COMPUTATION OF EXPECTED VALUES

\[
\log S_T = \log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)
\]

\[
E[f(S_T)] = E[f(\exp\{\log S_0 - \frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)\})]
\]

\[
= E[f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} N(0, 1)\})]
\]

\[
= \int_{-\infty}^{+\infty} f(S_0 \exp\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z\}) \phi(z) dz
\]

(7)

where \( \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2} z^2\} \)
IN PARTICULAR: $f(s) = s$:

\[
E[U] = E[S_T]
\]
\[
= \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z \right\} \phi(z) dz
\]
\[
= \left[ \frac{1}{\sqrt{2\pi}} \right] \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} \sigma^2 T + \sqrt{\sigma^2 T} z - \frac{1}{2} z^2 \right\} dz
\]
\[
= \left[ \frac{1}{\sqrt{2\pi}} \right] \int_{-\infty}^{+\infty} S_0 \exp\left\{-\frac{1}{2} (z - \sqrt{\sigma^2 T})^2 \right\} dz
\]
\[
= S_0 \int_{-\infty}^{+\infty} \phi(z - \sqrt{\sigma^2 T}) dz
\]
\[
= S_0 \int_{-\infty}^{+\infty} \phi(u) du \quad (u = z - \sqrt{\sigma^2 T})
\]
\[
= S_0
\]

IT FOLLOWS THAT EQUATION (5) IS SATISFIED
THE BLACK-SCHOLES-MERTON FORMULA

- THE OPTIONS PRICE FOR LARGE $n$ IS

$$E(\tilde{S}_T^{(\infty)} - e^{-rT}K)^+$$

- CAN COMPUTE IT EXPLICITELY USING EQUATION (7)

- THIS IS THE B-S-M FORMULA FOR THE PRICE OF A CALL OPTION

- YOU DON’T NEED TO USE A TREE IN THIS CASE
CONTINUOUS MARTINGALES

TWO CONTINUITIES:

• TIME ITSELF:

\[ M_t, \quad 0 \leq t \leq T \quad \text{(or } 0 \leq t < \infty) \]

• PROCESS PATH:

\[ t \rightarrow M_t = M_t(\omega) \quad \text{CONTINUOUS FUNCTION OF TIME} \]
(ADDITIVE) BROWNIAN MOTION $W_t$, $0 \leq t \leq T$

(1) $W_0 = 0$

(2) $t \to W_t(\omega)$ IS CONTINUOUS for each $\omega$

(3) HAS INDEPENDENT INCREMENTS

(4) $W_{t+s} - W_s \sim N(0, t)$

PICTURE OF (3):

\[
\begin{array}{c|c|c|c|c}
\Delta W_{t_0} & \Delta W_{t_1} & \Delta W_{t_2} & \text{INDEPENDENT} \\
= W_{t_1} - W_{t_0} & = W_{t_2} - W_{t_1} \\
0 = t_0 & t_1 & t_2 & t_3 & \text{ANY GRID}
\end{array}
\]

ADDITIVE PROPERTY (4):

$\Delta W_{t_0} \sim N(0, t_1)$, $\Delta W_{t_1} \sim N(0, t_2 - t_1)$

\[
\text{DELETE } t_1 : W_{t_2} - W_{t_0} = \frac{\Delta W_{t_0}}{N(0, t_1)} + \frac{\Delta W_{t_1}}{N(0, t_2 - t_1)} \underbrace{N(0, t_1) + N(0, t_2 - t_1)}_{\text{BY INDEP: } N(0, t_2)}
\]
\(\text{(3) + (4)} \implies W_t \text{ IS A MARTINGALE}\)

\[
E(W_{t+s} \mid \mathcal{F}_s) = E(W_{t+s} - W_s + W_s \mid \mathcal{F}_s)
\]
\[
= E(W_{t+s} - W_s \mid \mathcal{F}_s) + W_s
\]
\[
= \underbrace{E(W_{t+s} - W_s)}_{(\text{independence})} + W_s
\]
\[
= 0 \quad \text{since } W_{t+s} - W_s \sim N(0, t)
\]
\[
= W_s
\]
THE BLACK-SCHOLES MODEL: MULTIPLICATIVE
BROWNIAN MOTION

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_t - \frac{1}{2} \sigma^2 t) \]

EVOLUTION:

\[ \tilde{S}_t = \tilde{S}_0 \times \exp(\sigma W_u - \frac{1}{2} \sigma^2 u) \]
\[ \quad \times \exp(\sigma (W_t - W_u) - \frac{1}{2} \sigma^2 (t - u)) \]
\[ = \tilde{S}_u \times \exp(\sigma N(0, t - u) - \frac{1}{2} \sigma^2 (t - u)) \]
\[ = \tilde{S}_u \times \exp(\alpha Z - \frac{1}{2} \alpha^2) \quad \alpha^2 = \sigma^2 (t - u) \quad Z \sim N(0, 1) \]

MARTINGALE:

\[ E(\tilde{S}_t \mid \mathcal{F}_u) = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2) \mid \mathcal{F}_u) \]
\[ = \tilde{S}_u E(\exp(\sigma Z - \frac{1}{2} \sigma^2)) \text{ BY INDEPENDENCE} \]
\[ = \tilde{S}_u \times 1 \quad (\text{NORMAL}) \]
\[ = \tilde{S}_u \]
CLT FOR THE WHOLE PROCESS

\[ t_0 = 0 \quad t_1 = \frac{\sigma^2 T}{n} \quad t_2 = \frac{2\sigma^2 T}{n} \quad t_3 = \frac{3\sigma^2 T}{n} \quad t_k = \frac{k\sigma^2 T}{n} \]

STOCK PRICE PROCESS

\[ \log(\tilde{S}_t^{(n)}) - \log(S_0) = \sum_{t_i \leq t} X_i, \quad 0 \leq t \leq T \]

CONVERGENCE: AS \( n \to \infty \):

\[ \log(\tilde{S}_t^{(n)}) \xrightarrow{\mathcal{L}} \log(S_t) = \log(S_0) + \sigma W_t - \frac{1}{2} \sigma^2 t \]

GEOMETRIC BROWNIAN MOTION
APPLICATION TO OPTIONS

CONTINUOUS FUNCTIONALS

• $x = \{x_t, 0 \leq t \leq T\}$ A REALIZATION OF THE PROCESS

• $x \rightarrow h(x)$ TAKES REAL VALUES

• $x \rightarrow h(x)$ IS CONTINUOUS:

$$\sup_{0 \leq t \leq T} |x_t^{(n)} - x_t| \rightarrow 0 \quad \Rightarrow \quad h(x^{(n)}) \rightarrow h(x_t)$$

FOR $h$ CONTINUOUS:

$$h(\log(\tilde{S}_t^{(n)})) \xrightarrow{L} h(\log(\tilde{S}_t))$$

OR

$$h(\tilde{S}_t^{(n)}) \xrightarrow{L} h(\tilde{S}_t)$$

EXAMPLE OF MEANINGFUL LIMIT:

$$h(x) = \sup_{0 \leq t \leq T} x_t$$

LOOKBACK OPTIONS