MARTINGALES BASED ON IID:

ADDITIVE MG

\[ Y_1, \ldots, Y_t, \ldots: \text{IID } EY = 0 \]
\[ X_t = Y_1 + \ldots + Y_t \text{ is MG} \]

MULTIPLICATIVE MG

\[ Y_1, \ldots, Y_t, \ldots: \text{IID } EY = 1 \]
\[ X_t = Y_1 \times \ldots \times Y_t: \quad X_{t+1} = X_t Y_{t+1} \]

\[ E(X_{t+1} \mid \mathcal{F}_t) = E(X_t Y_{t+1} \mid \mathcal{F}_t) \]
\[ = X_t E(Y_{t+1} \mid \mathcal{F}_t) \]
\[ = X_t \times 1 \quad = X_t \]

MORE REALISTIC FOR STOCKS, FOR EXAMPLE

BINOMIAL MODEL: \( X_t = \tilde{S}_t, Y = e^{-r}u \text{ or } e^{-r}d \) under \( \pi \)
MARKOV PROCESSES

\( X_t, t = 0, \ldots, T \) is process adapted to filtration \((\mathcal{F}_t)\) 
\((X_t)\) is a Markov process if for every function \(f\) there is a function \(g\) so that
\[
E(f(X_{t+1}) \mid \mathcal{F}_t) = g(X_t)
\]
In other words: all current info relevant to the future of \((X_t)\) is contained in the current value of \((X_t)\).

FOR EXAMPLE:

\((S_t)\) IS MARKOV IN THE BINOMIAL TREE

\[
S_{t-1}(\omega_{t-1}) \quad \xleftarrow{\quad} \quad S_t(\omega_{t-1}, H) = uS_{t-1}(\omega_{t-1}) \quad S_t(\omega_{t-1}, T) = dS_{t-1}(\omega_{t-1})
\]

For \(S_{t-1}(\omega_{t-1}) = s_{t-1}\), and if \(Z = u\) with probability \(p(H)\) and \(= d\) with probability \(p(T)\)
\[
E(f(S_t) \mid \mathcal{F}_{t-1})(\omega_{t-1}) = E(f(ZS_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1})
\]
\[
= E(f(Zs_{t-1}) \mid \mathcal{F}_{t-1})(\omega_{t-1})
\]
\[
= E(f(Zs_{t-1})) \quad \text{(since } Z \text{ is independent of } \mathcal{F}_{t-1})
\]
\[
= g(s_{t-1})
\]
where:
\[
g(s) = E(f(Zs)) = f(us)p(H) + f(ds)p(T)
\]
This works for both actual and risk neutral probability
General principle for taking out what is known

Lemma: Let $X = (X_1, ..., X_K)$ and $Y = (Y_1, ..., X_L)$ be random variables, and let $f(x, y)$ be a function. If $X = (X_1, ..., X_p)$ is $\mathcal{G}$-measurable, then on the set $\omega \in \{X = x\}$

$$E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(x, Y) \mid \mathcal{G})(\omega).$$

Proof of Lemma: Let $\mathcal{P}$ be the partition corresponding to $\mathcal{G}$. For each $\omega \in \{X = x\}$, there is a $B \in \mathcal{P}$, so that $\omega \in B \subseteq \{X = x\}$. (This is since $X$ is $\mathcal{G}$-measurable.) Then

$$E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(X, Y) \mid B)$$

$$= E(f(x, Y) \mid B) \text{ since } X(\omega) = x \text{ on } B$$

$$= E(f(x, Y) \mid \mathcal{G})(\omega)$$

QED

When $Y$ is independent of $\mathcal{G}$

$$E(f(x, Y) \mid \mathcal{G})(\omega) = E(f(x, Y))$$

consequently

$$E(f(X, Y) \mid \mathcal{G})(\omega) = E(f(x, Y))$$
THE STRUCTURE OF EUROPEAN OPTIONS IN THE BINOMIAL TREE

Payoff $V = v_T(S_T)$.

Induction: assume value of option at $t+1$ is $v_{t+1}(S_{t+1})$. Then

\[
\text{value of option at } t = E_\pi (e^{-r} v_{t+1}(S_{t+1}) \mid \mathcal{F}_t) \\
= E_\pi (e^{-r} v_{t+1}(Z S_t) \mid \mathcal{F}_t) \\
\text{where } Z = u \text{ or } = d \\
= v_t(S_t)
\]

where

\[
v_t(s) = E_\pi v_{t+1}(Z s) = \pi(H)v_{t+1}(us) + \pi(T)v_{t+1}(ds)
\]
A NON-MARKOV PROCESS

\[ M_t = \max_{0 \leq u \leq t} S_u \]

Fig. 2.5.1. The maximum stock price to date.

(from Shreve p. 48)
CREATING A MARKOV PROCESS
BY ADDING STATE VARIABLES

\[ M_t = \max_{0 \leq u \leq t} S_u \]

CLAIM: \((S_t, M_t)\) is a Markov process

Proof: If \(S_t = s\) and \(M_t = m\): with \(Z = u\) or \(d\):

\[ S_{t+1} = sZ \quad \text{and} \quad M_{t+1} = M_t \vee S_{t+1} = m \vee sZ \]

\((x \vee y = \max(x, y) \text{ and } x \wedge y = \min(x, y))\)

\[ E(f(S_{t+1}, M_{t+1}) \mid F_t) = E(f(sZ, m \vee (sZ))) \]

\[ = g(s, m) \]

by appropriate definition of \(g\) \hfill QED

It follows, by induction, that, for payoff \(v_T(S_T, M_T)\):

\[ \text{value of option at } t = v_t(S_t, M_t) \]

General Theorem (Feynman-Kac): If \(X_0, X_1, \ldots, X_T\) is a Markov process under \(\pi\) (e.g. \(X_t = (S_t, M_t)\)). For payoff \(v_T(X_T)\),

\[ \text{value of option at } t = v_t(X_t) \]

for some function \(v_t(x)\).
CHANGE OF MEASURE

$Q, R$: PROBABILITIES, $\mathcal{G}$: $\sigma$-field

$R$ IS ABSOLUTELY CONTINUOUS UNDER $Q$ (on $\mathcal{G}$):

\[ R \ll Q : \forall A \in \mathcal{G} : Q(A) = 0 \Rightarrow P(A) = 0 \]

$R$ IS EQUIVALENT TO $Q$ (on $\mathcal{G}$):

\[ R \sim Q : R \ll Q \text{ AND } Q \ll R \]

ACTUAL AND RISK NEUTRAL:

NO ARBITRAGE $\Rightarrow P \sim \pi$

BECAUSE

(i) If $\pi(A) > 0$, $P(A) = 0$

\[ S_T = I_A = \begin{cases} 1 & \text{if } A \\ 0 & \text{if not} \end{cases} \quad \text{so } P(S_T = 0) = 1 \]

But $\pi(S_T > 0) > 0 \Rightarrow \pi\left(\frac{1}{B_T} S_T > 0\right) > 0$

$\Rightarrow S_0 = E_\pi \frac{1}{B_T} S_T > 0$ ARBITRAGE

(ii) If $\pi(A) = 0$, $P(A) > 0$:

\[ P(S_T > 0) > 0 \]

\[ S_0 = E_\pi \frac{1}{B_T} S_T = 0 \quad \text{ARBITRAGE} \]
BOND WIH DEFAULT

NUMERAIRE: $B_t = e^{rt}$  $t = 0, 1$

BOND: $V_t = 1$ for $t = 0$

$$= \begin{cases} e^m & \text{no default} \\ 0 & \text{default} \end{cases} \quad \text{for } t = 1$$

$$\tilde{V}_t = \frac{V_t}{B_t} = 1 \text{ for } t = 0$$

$$= \begin{cases} e^{m-r} & \text{no default} \\ 0 & \text{default} \end{cases} \quad \text{for } t = 1$$

$$1 = \tilde{V}_0 = E_\pi \tilde{V}_1 = e^{m-r}\pi( \text{ no default} ) \quad (+0 \times \pi( \text{ default} ))$$

Condition: $m > r$ or $m \geq r$? or ??

$$P(\text{default}) = 0 \Rightarrow \pi(\text{default}) = 0 \quad (\pi \ll P)$$

$$\Rightarrow \pi(\text{no default}) = 1$$

$$\Rightarrow m = r$$

$$1 > P(\text{default}) > 0 \Rightarrow 1 > \pi(\text{default}) > 0$$

$$\Rightarrow m > r$$
RADON-NIKODYM DERIVATIVES

$Q, R$: PROBABILITIES, $Q \ll R$ (on $\mathcal{G}$)

$\mathcal{P}$ is partition associated with $\mathcal{G}$

DEFINITION OF R-N DERIVATIVE

$$\frac{dR}{dQ}(\omega) = \frac{R(B)}{Q(B)} \text{ when } \omega \in B \in \mathcal{P}$$

CAVEAT: $\frac{dR}{dQ}$ DEPENDS ON $\mathcal{G}$: $\frac{dR}{dQ} = \frac{dR}{dQ} |_{\mathcal{G}}$

PROPERTIES

- $Q(\frac{dR}{dQ} \geq 0) = 1$
- If $R \sim Q$: $Q(\frac{dR}{dQ} > 0) = 1$
- $E_Q \left( \frac{dR}{dQ} \right) = 1$
- For all $\mathcal{G}$-measurable $Y$: $E_R (Y) = E_Q \left( Y \frac{dR}{dQ} \right)$
- If $R \sim Q$: $\frac{dQ}{dR} = \left( \frac{dR}{dQ} \right)^{-1}$
EXAMPLE OF PROOF

Since $Y$ is $\mathcal{G}$-measurable:

for every $B \in \mathcal{P}$: $Y(\omega)$ is constant for $\omega \in B$: $Y(\omega) = Y(B)$. Hence:

$$E_R(Y) = \sum_{B \in \mathcal{P}} Y(B)R(B)$$

$$= \sum_{B \in \mathcal{P}} Y(B) \frac{dR}{dQ}(B)Q(B)$$

$$= E_Q \left( Y \frac{dR}{dQ} \right)$$
BINOMIAL TREES

\[ S_2 = u^2 S_0 \quad \text{H, then H} \]

\[ S_2 = udS_0 \quad \text{H, then T} \]

\[ S_2 = d^2 S_0 \quad \text{T, then H} \]

\[ S_2 = d^2 S_0 \quad \text{T, then T} \]

ACTUAL MEASURE: \( P(H), P(T) \)

RISK NEUTRAL MEASURE: \( \pi(H), \pi(T) \)

\( \sigma \)-FIELD \( \mathcal{F}_n \) OR PARTITION \( \mathcal{P}_n \): BASED ON \( n \) FIRST COIN TOSSES: \( (\omega_1, \ldots, \omega_n) \)
CALCULATION OF R-N DERIVATIVE IN BINOMIAL CASE

- For every sequence $\omega = (\omega_1, \ldots, \omega_n)$: $\{\omega\} = \{ (\omega_1, \ldots, \omega_n) \}$ is a set in $P_n$

- Value of derivative for this set:

$$
\frac{d\pi}{dP}(\omega_1, \ldots, \omega_n) = \frac{\pi(\omega_1, \ldots, \omega_n)}{P(\omega_1, \ldots, \omega_n)}
= \frac{\pi(\omega_1) \times \cdots \times \pi(\omega_n)}{P(\omega_1) \times \cdots \times P(\omega_n)}
= \left( \frac{\pi(H)}{P(H)} \right)^H \left( \frac{\pi(T)}{P(T)} \right)^T
$$

where $H = H(\omega) =$ # of heads among $\omega_1, \ldots, \omega_n$

AMBIGUITY ABOUT "\omega"

Could consider $\omega = (\omega_1, \ldots, \omega_n, \ldots)$ (the outcome of infinitely many coin tosses). Then

$$
\frac{d\pi}{dP} |_{\mathcal{F}_n} (\omega) = \frac{d\pi}{dP}(\omega_1, \ldots, \omega_n)
$$

Each set in $P_n$: one $(\omega_1, \ldots, \omega_n)$, many $(\omega_1, \ldots, \omega_n, \ldots)$
RELATIONSHIP BETWEEN R-N DERIVATIVE AND STOCK PRICE IN BINOMIAL CASE

\[ S_n = S_0 \times u^{#H} \times d^{#T} \]

\[ = S_0 d^n \times \left( \frac{u}{d} \right)^{#H} \]

since \( #H + #T = n \). Thus:

\[ #H = \frac{\log S_n - \log S_0 - n \log d}{\log u - \log d} \]

ON THE OTHER HAND:

\[ \frac{d\pi}{dP}(\omega) = \left( \frac{\pi(H)}{P(H)} \right)^{#H} \left( \frac{\pi(T)}{P(T)} \right)^{#T} \]

\[ = f(S_n, n) \]

\[ \frac{d\pi}{dP} \text{ IS A FUNCTION OF } S_n \]
STATE PRICE DENSITY

For payoffs at time $t$:

$$\xi_t(\omega) = e^{-rt} \frac{d\pi}{dP} |_{\mathcal{F}_t} (\omega)$$

Price at time 0 for payoff $I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{otherwise} \end{cases}$

at time $t$:

$$e^{-rt} E_\pi (I_A) = e^{-rt} E_P \left( I_A \frac{d\pi}{dP} \right)$$

$$= E_P (I_A \xi_t)$$

State price corresponding to $\omega$: $A = \{\omega\}$:

$$E_P (I_{\{\omega\}} \xi_t) = \xi_t(\omega) P\{\omega\}$$

General valuation formula:

price for payoff $V$ at time $t = e^{-rt} E_\pi (V) = E_P (V \xi_t)$
THE RADON-NIKODYM DERIVATIVE  
IS A MARTINGALE  

DEFINE $Z_t = \frac{d\pi}{dP} |_{\mathcal{F}_t}$  

IN THE BINOMIAL CASE:  

$Z_{t+1} = Z_t Y_{t+1}$  

WHERE $Y_1, Y_2, \ldots$ ARE IID,  

$Y_t = \left\{ \begin{array}{ll} \frac{\pi(H)}{P(H)} & \text{if outcome H at time } t \\ \frac{\pi(T)}{P(T)} & \text{if outcome T at time } t \end{array} \right.$  

AND  

$E_P(Y) = \frac{\pi(H)}{P(H)} P(H) + \frac{\pi(T)}{P(T)} P(T) = \pi(H) + \pi(T) = 1$  

$Z_0 = 1$  

A TYPICAL MULTIPLICATIVE MARTINGALE
THE GENERAL CASE

\[ Z_t = \frac{d\pi}{dP} |_{\mathcal{F}_t} \] 

Filtration on partition form: \( \mathcal{P}_t \)

IF \( A \in \mathcal{P}_t \) AND \( A = \cup_q B_q \) WHERE \( B_q \in \mathcal{P}_{t+1} \):

\[
E(Z_{t+1} \mid A) = \sum_q Z_{t+1}(B_q) P(B_q \mid A)
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} P(B_q \mid A)
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q \cap A)}{P(A)}
\]

\[
= \sum_q \frac{\pi(B_q)}{P(B_q)} \frac{P(B_q)}{P(A)} \text{ since } B_q \subseteq A
\]

\[
= \frac{1}{P(A)} \sum_q \pi(B_q)
\]

\[
= \frac{\pi(A)}{P(A)}
\]

\[
= Z_t(A)
\]

IF \( \omega \in A \):

\[
E(Z_{t+1} \mid \mathcal{P}_t)(\omega) = E(Z_{t+1} \mid A) = Z_t(A) = Z_t(\omega)
\]

THUS: \((Z_t)\) IS A MARTINGALE

WInter 2006

Per A. Mykland
THE R-N DERIVATIVE REPRESENTATION BY FINAL VALUE

IF $T$ IS THE FINAL TIME, AND $\frac{d\pi}{dP}$ IS R-N DERIVATIVE ON $\mathcal{F}_T$:

$$Z_t = E (Z_T \mid \mathcal{F}_t) = E_P \left( \frac{d\pi}{dP} \mid \mathcal{F}_t \right) \text{ for } t \leq T$$
THE R-N DERIVATIVE:  
CONDITIONAL EXPECTATIONS

**Theorem:** Suppose $Q \ll P$ on $\mathcal{F}$. Let $dQ/dP \otimes \mathcal{F}$ and $\mathcal{G} \subseteq \mathcal{F}$. Then

$$E_Q(X \mid \mathcal{G}) = \frac{E_P \left( X \frac{dQ}{dP} \mid \mathcal{G} \right)}{E_P \left( \frac{dQ}{dP} \mid \mathcal{G} \right)}.$$

**IN THE TIME DEPENDENT SYSTEM:** $\mathcal{G} = \mathcal{F}_t$

$$E_{\pi}(X \mid \mathcal{F}_t) = \frac{E_P \left( X Z_T \mid \mathcal{F}_t \right)}{E_P \left( Z_T \mid \mathcal{F}_t \right)} = \frac{E_P \left( X Z_T \mid \mathcal{F}_t \right)}{Z_t}.$$

**PRICE AT TIME $t$ OF PAYOFF $V$ AT TIME $T$:**

$$\tilde{V}_t = E_{\pi}(\tilde{V} \mid \mathcal{F}_t) = \frac{E_P \left( \tilde{V} Z_T \mid \mathcal{F}_t \right)}{Z_t}$$

$$V_t = \frac{E_P \left( e^{-rT} V Z_T \mid \mathcal{F}_t \right)}{e^{-rt} Z_t} = \frac{E_P \left( V \xi_T \mid \mathcal{F}_t \right)}{\xi_t}.$$
ANOTHER APPLICATION OF
THE CONDITIONAL EXPECTATION FORMULA:

MODIFYING PATH DEPENDENT OPTIONS
IN THE BINOMIAL MODEL

\( V = \text{payoff of option (at } T) \)

\[
V' = E_\pi (V \mid S_T)
\]

\[
= \frac{E(\frac{d\pi}{dP} \cdot V \mid S_T)}{E(\frac{d\pi}{dP} \mid S_T)} \quad (\mathcal{G} = \sigma(S_T))
\]

\[
= \frac{d\pi}{dP} E(\frac{d\pi}{dP} \mid S_T) \quad \text{(since } \frac{d\pi}{dP} \text{ is a function of } S_T) \]

\[
= E(V \mid S_T)
\]

WE HAVE HERE USED \( \mathcal{G} = \sigma(S_T) \)
ARE PATH DEPENDENT OPTIONS OPTIMAL?

\[ V = \text{payoff} \]
\[ V' = \mathbb{E}_\pi (V \mid S_T) = \mathbb{E}(V \mid S_T) \]

- Price:

  \[
  \text{Price of } V = e^{-rT} \mathbb{E}_\pi V \\
  = e^{-rT} \mathbb{E}_\pi V' \quad \text{tower property} \\
  = \text{price of } V'
  \]

- Expected payoff of \( V \):

  \[
  e^{-rT} \mathbb{E}V = e^{-rT} \mathbb{E}V' \quad \text{tower property} \\
  = \text{expected payoff of } V'
  \]

- The Rao-Blackwell inequality:

  \[
  \text{Var} \ (V) = \mathbb{E} \left( \text{Var} \ (V \mid S_T) \right) + \text{Var} \ (\mathbb{E}(V \mid S_T)) \\
  = \mathbb{E} \left( \text{Var} \ (V \mid S_T) \right) + \text{Var} \ (V') \\
  > \text{Var} \ (V')
  \]

  unless \( V = V' \).
UTILITY: TYPICAL ASSUMPTIONS

- more is better: $V_1 \geq V_2 \implies U(V_1) \geq U(V_2)$
- risk aversion: strict concavity:

$$U(\alpha V_1 + (1 - \alpha)V_2) > \alpha U(V_1) + (1 - \alpha)U(V_2)$$

non-strict concavity: replace “<” by “≤”

Jensen: $U$ strictly concave: unless $V$ is $\mathcal{G}$-measurable:

$$E(U(V) \mid \mathcal{G}) < U(E(V) \mid \mathcal{G})) \quad P\text{-a.s.}$$

PATH DEPENDENT OPTIONS AGAIN

$$V' = E_{\pi}(V \mid S_T) = E(V \mid S_T)$$

- Price of $V = \text{price of } V'$

$$EU(V) = EE(U(V) \mid S_T)$$

$$< EU(E(V \mid S_T)) \quad \text{unless } V \in \sigma(S_T)$$

$$= EU(V')$$
OPTIMAL INVESTMENT IN BIN. MODEL

MAXIMIZATION OF UTILITY SUBJECT TO INITIAL CAPITAL:

$$\max EU(V_T)$$

subject to constraints:

- capital constraint: $$V_0 = v$$
- replicability: $$\tilde{V}_T = V_0 + \sum_{t=0}^{T-1} \Delta_t \Delta \tilde{S}_t$$

BY COMPLETENESS: CONSTRAINTS EQUIVALENT TO (with $$\xi_T = e^{-rT}(d\pi/dP)$$)

$$E_{\pi} \tilde{V}_T = v$$

or

$$EV_T \xi_T = v$$

BY ARGUMENT ON PREVIOUS PAGE: $$V_T$$ IF FUNCTION OF $$S_T$$, OR, EQUIVALENTLY, $$\xi_T$$:

$$V_T = f(\xi_T)$$
REFORMULATION OF PROBLEM

\( \xi_T \) can take values \( x_1, \ldots, x_{T+1} \) Problem becomes:

\[
\max_f \sum U(f(x_i))P(\xi = x_i)
\]

subject to:

\[
\sum f(x_i)x_iP(\xi = x_i) = v
\]

Lagrangian:

\[
L = \sum U(f(x_i))P(\xi = x_i) - \lambda \left( \sum f(x_i)x_iP(\xi = x_i) - v \right)
\]

Want:

\[
0 = \frac{\partial L}{\partial f(x_i)} = U'(f(x_i))P(\xi = x_i) - \lambda x_iP(\xi = x_i)
\]

In other words:

\[
U'(f(x_i)) = \lambda x_i
\]

or:

\[
U'(V_T) = \lambda \xi_T
\]

Finally:

\[
V_T = (U')^{-1}(\lambda \xi_T)
\]