Uncertain Volatility and Interest

\( \Lambda_t = \) zero coupon bond, \( \Lambda_T = 1 \)

\( \tilde{S}_t = S_t / \Lambda_t \)

\[ d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t \]

Suppose

\[ \sigma^- \leq \sigma_t \leq \sigma^+ \quad \text{for all } t \quad (1) \]

(Pointwise bound assumption) or

\[ \equiv^- \leq \int_0^T \sigma_t^2 dt \leq \equiv^+ \quad (2) \]

(Integral bound assumption)

What is the “ask” price \( A \) for European payoff

\[ \eta = f(S_T) = f(\tilde{S}_T)? \quad (\text{note: } \Lambda_T = 1 : S_T = \tilde{S}_T) \]
Case: Pointwise bounds

\[ \sigma^- \leq \sigma_t \leq \sigma^+ \]  

(Avellaneda, Levy, Paras (1995), Lyons (1995)).

Let \( V(s, t) \) solve \((V(\tilde{S}_t, t) = \text{discounted value of portfolio})\)

\[
\begin{cases}
  V_t(s, t) + \frac{1}{2} s^2 \max(\sigma^2 V_{SS}(s, t)) = 0 \\
  V(s, T) = f(s)
\end{cases}
\]

(3): Barenblatt equation

Ito + (3)

\[
dV(\tilde{S}_t, t) = V_s(\tilde{S}, t)d\tilde{S}_t + V_t(\tilde{S}, t)dt + \frac{1}{2} \tilde{S}_t^2 \sigma_t^2 V_{SS}(\tilde{S}_t, t)dt
\]

\[
= V_S(\tilde{S}, t)d\tilde{S}_t - d\tilde{D}_t
\]

\[
d\tilde{D}_t = -(V_t(\tilde{S}, t) + \frac{1}{2} \tilde{S}_t \sigma_t^2 V_{SS}(\tilde{S}, t))dt
\]

\[
= \frac{1}{2} \tilde{S}_t^2 \max(\sigma^2 V_{SS}(\tilde{S}_t, t)) - \sigma_t^2 V_{SS}(\tilde{S}_t, t))dt
\]

\[ \geq 0 \]
Interpretation 1:
\[ V(\tilde{S}_t, t) = \text{super-replication of } f(S_T) \]
\[ D_t = \text{dividend} \]

Interpretation 2:
\[ V(\tilde{S}_t, t) = P^* - \text{supermartingale} \]
\[ V(\tilde{S}_T, T) = f(\tilde{S}_T) \]

In any case: \( \tilde{A} \leq V(\tilde{S}_0, 0) \)

However, if
\[ \sigma_t^2 = \begin{cases} 
(\sigma^+)^2 & \text{if } V_{SS}(\tilde{S}_t, t) \geq 0 \\
(\sigma^-)^2 & \text{if } V_{SS}(\tilde{S}_t, t) < 0 
\end{cases} \]

(\text{one choice of } P^* \text{ under (1)})

then \( D_t \equiv 0 \), so \( V(\tilde{S}_t, t) \) exact replication on discounted scale,

so \( \tilde{A} \geq V(\tilde{S}_0, 0) \)

Conclusion:
\[ \tilde{A} = V(\tilde{S}_0, 0) \]

\( f \) convex: use \( V(s, t) = \text{BS with } \sigma = \sigma^+ \)
\( f \) concave: use \( V(s, t) = \text{BS with } \sigma = \sigma^- \)
Case: Integral bounds (2)

Recall:

$$\Lambda_t = \text{zero coupon bond, maturity } T$$

$$\tilde{S}_t = S_t / \Lambda_t$$

$$d\tilde{S}_t = \mu_t \tilde{S}_t dt + \sigma_t \tilde{S}_t dW_t$$

Suppose

$$\equiv^- \leq \int_0^T \sigma_u^2 du \leq \equiv^+ \tag{2}$$

What is the ask price $A$ for payoff?

$$\eta = f(S_T) = f(\tilde{S}_T)$$

$$A = \sup_{P^* \sim (2)} E^* \Lambda_0 f(\tilde{S}_T)$$

Computation of $A$:

Under $P^* \sim (2)$:

$$d\tilde{S}_t = \sigma_t \tilde{S}_t dW_t^*$$

Or

$$d \log \tilde{S}_t = -\frac{\sigma_t^2}{2} dt + \sigma_t dW_t^*$$
We set
\[ \tau_t = \int_0^t \sigma_u^2 \, du \]
(requiring $t \to \tau_t$ to be 1-1 ($\sigma_u^2 > 0 \text{ a.s.}$)) and define $\tilde{S}$ by
\[ \tilde{S}_t = \tilde{S}_{\tau_t} \]
Define $M_t = \int_0^t \sigma_u dW_u^* = \hat{M}_{\tau_t}$

If $\mathcal{F}_t = \hat{\mathcal{F}}_{\tau_t}$, then
$\hat{M}_t$ is $(\hat{\mathcal{F}}_t) - \text{MG}$ (show below)

Also
\[ [\hat{M}, \hat{M}]_{\tau_t} = [M, M]_t = \int_0^t \sigma_u^2 \, du = \tau_t \]
so
\[ [\hat{M}, \hat{M}]_u = u \Rightarrow \hat{M}_t \text{ is std BM } \cdot \hat{\mathcal{F}}_t \]

\[ \log \tilde{S}_{\tau_t} = \log \tilde{S}_t \]
\[ = \log \tilde{S}_0 - \int_0^t \frac{\sigma_u^2}{2} \, du + \int_0^t \sigma_u dW_u^* \]
\[ = \log \tilde{S}_0 - \frac{1}{2} \tau_t + \hat{M}_{\tau_t} \]
or
\[ \log \hat{S}_t = \log \tilde{S}_0 - \frac{1}{2} t + \hat{M}_t \]
or
\[ d\hat{S}_t = \hat{S}_t d\hat{M}_t \]
Why is $(\hat{M}_t)$ an $(\hat{\mathcal{F}}_t) -$ MG?

Set $\lambda_t: \tau_{\lambda_t} = \tau_{\tau_t} = t$

If $t \rightarrow \tau_t$ is 1 - 1 then $\lambda_t$ is continuous

since $M_t = \hat{M}_{\tau_t}$, get $\hat{M}_t = M_{\lambda_t}$

by $\mathcal{F}_t = \hat{\mathcal{F}}_{\tau_t}$ we mean $\hat{\mathcal{F}}_t = \mathcal{F}_{\lambda_t}$

Use optional stopping:

$$E(\hat{M}_t | \hat{\mathcal{F}}_s) = E(M_{\lambda_t} | \mathcal{F}_{\lambda_s})$$
$$= M_{\lambda_s}$$
$$= \hat{M}_s$$

Recall

$$\mathcal{F}_{\lambda} = \{ A \in \mathcal{F}: \forall t, \ A \cap \{ \lambda \leq t \} \in \mathcal{F}_t \}$$
Back to original problem:

\[ \tilde{S}_T = \tilde{S}_{\tau_T} \quad \tau_T = \int_0^T \sigma_u^2 du \]

\[ P^* \sim (2) \iff \Xi^- \leq \int_0^T \sigma_u^2 du \leq \Xi^+ \]

\[ \iff \Xi^- \leq \tau_T \leq \Xi^+ \]

It follows:

\[ A = \sup_{P^* \sim (2)} \Lambda_0 E^* f(\tilde{S}_T) \]

\[ = \sup_{\Xi^- \leq \tau_T \leq \Xi^+} \Lambda_0 E^* f(\tilde{S}_{\tau_T}) \]

Also, \( \tilde{S} \) is the same for all \( P^* \)

\[ \tilde{S}_0 = \tilde{S}_0 = \frac{1}{\Lambda_0} S_0 \quad d\tilde{S}_t = \tilde{S}_t d\tilde{M}_t \]

If \( f = \text{convex}, \) Jensen’s inequality \( \Rightarrow \)

\[ E[f(\tilde{S}_{\Xi^+}) | \tilde{F}_{\tau_T}] \geq f(\tilde{S}_{\tau_T}) \quad \text{since} \ \tau_T \leq \Xi^+ \]

so

\[ A = \Lambda_0 E^+ f(\tilde{S}_{\Xi^+}) = BS(S_0, - \log \Lambda_0, \Xi^+) \]

If \( f = \text{concave}: \)

\[ E[f(\tilde{S}_{\tau_T}) | \mathcal{F}_{\Xi^-}] \leq f(\tilde{S}_{\Xi^-}) \]

\[ A = \Lambda_0 E^* f(\tilde{S}_{\Xi^-}) = BS(S_0, - \log \Lambda_0, \Xi^-) \]
Integral bound + traded options?

Ask price $A$ for payoff $f(\tilde{S}_T) = (\tilde{S}_T - K)^+$ under call prices

$$\Lambda_0 E^*(\tilde{S}_T - K_i)^+ = C_0^{(i)} \quad i = 1, \ldots, p$$

(5)

Set

$$h_\lambda(s) = (s - K)^+ + \sum_{i=1}^{P} \lambda_i [(s - K_i)^+ - \tilde{C}_0^{(i)}]$$

$$A \leq \inf_{\lambda} \sup_{P^* \sim (2)} \Lambda_0 E^* h_\lambda(\tilde{S}_T)$$

since one can replicate $h_\lambda(\tilde{S}_T)$ instead of $f(\tilde{S}_T)$. Any $\lambda \Rightarrow$ sup over $P^*$ satisfying (5):

$$A \leq \sup_{P^* \sim (2),(5)} \Lambda_0 E^* (\tilde{S}_T - K)^+$$

Some regularity conditions: $A = \text{this supremum}$
Time change:

\[ A = \sup_\tau \Lambda_0 E^*(\hat{S}_\tau - K)^+ \]

where \( \tau = \) stopping time:

\[ \Xi^- \leq \tau \leq \Xi^+ \]  \hspace{1cm} (2)’

\[ \Lambda_0 E^*(\hat{S}_\tau - K_i)^+ = C_0^{(i)} \]  \hspace{1cm} (5)’

\( \hat{S} \) as before: \( d\hat{S}_t = \hat{S}_td\hat{M}_t \)

If \( \tau' \) solves problem (in place of \( \tau \)), set

\[ \tau = \inf\{t \geq \tau' \mid \hat{S}_t = K_i\} \wedge \Xi^+ , \]

then

\[ E^*[(\hat{S}_\tau - K_i)^+ \mid \mathcal{F}_{\tau'}] = (S_{\tau'} - K_i)^+ \]

and (Jensen)

\[ E^*(\hat{S}_\tau - K)^+ \geq E^*(\hat{S}_{\tau'} - K)^+ \]

Can take

\[ \hat{S}_\tau = K_i \quad \text{for some } i, \text{ or } \tau = \Xi^+ \]
Simple case: all calls have some implied volatility \( \equiv \):

\[
BS(S_0, -\log \Lambda_0, \equiv) = \Lambda_0 E^*(\hat{S}_\equiv - K_i)^+ = C^{(i)}_0
\]

Can take

\[
\tau' = \equiv
\]

This gives optimal \( A \).