Uncertain Volatility and Interest

$$\Lambda_t = \text{ zero coupon bond, } \Lambda_T = 1$$

$$\widetilde{S}_t = S_t / \Lambda_t$$

$$d\widetilde{S}_t = \mu_t \widetilde{S}_t dt + \sigma_t \widetilde{S}_t dW_t$$

Suppose

$$\sigma^- \leq \sigma_t \leq \sigma^+ \text{ for all } t \tag{1}$$

(Pointwise bound assumption) or

$$\Xi^- \leq \int_0^T \sigma_t^2 dt \leq \Xi^+ \tag{2}$$

(Integral bound assumption)

What is the "ask" price A for European payoff

$$\eta = f(S_T) = f(\widetilde{S}_T)? \quad (\text{note: } \Lambda_T = 1 : S_T = \widetilde{S}_T)$$

Case: Pointwise bounds

$$\sigma^- \leq \sigma_t \leq \sigma^+ \tag{1}$$

(Avellaneda, Levy, Paras (1995), Lyons (1995)).

Let $V(s,t)$ solve ($V(\widetilde{S}_t, t)$ = discounted value of portfolio)

$$\begin{cases} V_t(s,t) + \frac{1}{2}s^2 \max\limits_{(1)}(\sigma^2 V_{SS}(s,t)) = 0 & (3) \\ V(s,T) = f(s) & (4) \end{cases}$$

(3): Barenblatt equation

Ito + (3)

$$\begin{aligned} dV(\widetilde{S}_t, t) &= V_s(\widetilde{S}, t)d\widetilde{S}_t + V_t(\widetilde{S}, t)dt + \tfrac{1}{2}\widetilde{S}_t^2 \sigma_t^2 V_{SS}(\widetilde{S}_t, t)dt \\ &= V_S(\widetilde{S}, t)d\widetilde{S}_t - d\widetilde{D}_t \end{aligned}$$

$$\begin{aligned} d\widetilde{D}_t &= -(V_t(\widetilde{S}, t) + \tfrac{1}{2}\widetilde{S}_t \sigma_t^2 V_{SS}(\widetilde{S}, t))dt \\ &= \tfrac{1}{2}\widetilde{S}_t^2(\max_{(1)}[\sigma^2 V_{SS}(\widetilde{S}_t, t)] - \sigma_t^2 V_{SS}(\widetilde{S}_t, t))dt \\ &\geq 0 \end{aligned}$$

Interpretation 1:

$$\begin{aligned} V(\widetilde{S}_t, t) &= \text{ super-replication of } f(S_T) \\ D_t &= \text{ dividend} \end{aligned}$$

Interpretation 2:

$$\begin{aligned} V(\widetilde{S}_t, t) &= P^* - \text{ supermartingale} \\ V(\widetilde{S}_T, T) &= f(\widetilde{S}_T) \end{aligned}$$

In any case: $\widetilde{A} \leq V(\widetilde{S}_0, 0)$

However, if

$$\sigma_t^2 = \begin{cases} (\sigma^+)^2 & \text{if } V_{SS}(\widetilde{S}_t, t) \geq 0 \\ (\sigma^-)^2 & \text{if } V_{SS}(\widetilde{S}_t, t) < 0 \end{cases}$$

(one choice of P^* under (1))

then $D_t \equiv 0$, so $V(\widetilde{S}_t, t)$ exact replication on discounted scale,

so $\widetilde{A} \geq V(\widetilde{S}_0, 0)$

Conclusion:

$$\widetilde{A} = V(\widetilde{S}_0, 0)$$

$$f \text{ convex: use } V(s, t) = \text{ BS with } \sigma = \sigma^+$$

$$f \text{ concave: use } V(s, t) = \text{ BS with } \sigma = \sigma^-$$

Case: Integral bounds (2)

Recall:

$\Lambda_t =$ zero coupon bond, maturity T

$\widetilde{S}_t = S_t/\Lambda_t$

$$d\widetilde{S}_t = \mu_t \widetilde{S}_t dt + \sigma_t \widetilde{S}_t dW_t$$

Suppose

$$\Xi^- \leq \int_0^T \sigma_u^2 du \leq \Xi^+ \tag{2}$$

What is the ask price A for payoff?

$$\eta = f(S_T) = f(\widetilde{S}_T)$$

$$A = \sup_{P^* \sim (2)} E^* \Lambda_0 f(\widetilde{S}_T)$$

Computation of A:

Under $P^* \sim (2)$:

$$d\widetilde{S}_t = \sigma_t \widetilde{S}_t dW_t^*$$

Or

$$d \log \widetilde{S}_t = -\frac{\sigma_t^2}{2} dt + \sigma_t dW_t^*$$

We set

$$\tau_t = \int_0^t \sigma_u^2 du$$

(require $t \to \tau_t$ to be 1-1 ($\sigma_u^2 > 0$ $a.s.$)) and define $\widehat{S}$ by

$$\widetilde{S}_t = \widehat{S}_{\tau_t}$$

Define $M_t = \int_0^t \sigma_u dW_u^* = \widehat{M}_{\tau_t}$

If $\mathcal{F}_t = \widehat{\mathcal{F}}_{\tau_t}$, then

$$\widehat{M}_t \text{ is } (\widehat{\mathcal{F}}_t) - \text{ MG} \quad \text{(show below)}$$

Also

$$[\widehat{M}, \widehat{M}]_{\tau_t} = [M, M]_t = \int_0^t \sigma_u^2 du = \tau_t$$

so

$$[\widehat{M}, \widehat{M}]_u = u \Rightarrow \widehat{M}_t \text{ is std BM w.r.t. } (\widehat{\mathcal{F}}_t)$$

$$\begin{aligned} \log \widehat{S}_{\tau_t} &= \log \widetilde{S}_t \\ &= \log \widetilde{S}_0 - \int_0^t \frac{\sigma_u^2}{2} du + \int_0^t \sigma_u dW_u^* \\ &= \log \widetilde{S}_0 - \tfrac{1}{2}\tau_t + \widehat{M}_{\tau_t} \end{aligned}$$

or

$$\log \widehat{S}_t = \log \widetilde{S}_0 - \tfrac{1}{2}t + \widehat{M}_t$$

or

$$d\widehat{S}_t = \widehat{S}_t d\widehat{M}_t$$

Why is $(\widehat{M}_t)$ an $(\widehat{\mathcal{F}}_t)-$ MG?

Set λ_t: $\tau_{\lambda_t} = \lambda_{\tau_t} = t$

If $t \to \tau_t$ is $1-1$
then λ_t is continuous

since $M_t = \widehat{M}_{\tau_t}$, get $\widehat{M}_t = M_{\lambda_t}$

by $\mathcal{F}_t = \widehat{\mathcal{F}}_{\tau_t}$ *we mean* $\widehat{\mathcal{F}}_t = \mathcal{F}_{\lambda_t}$

Use optional stopping:

$$\begin{aligned} E(\widehat{M}_t \mid \widehat{\mathcal{F}}_s) &= E(M_{\lambda_t} \mid \mathcal{F}_{\lambda_s}) \\ &= M_{\lambda_s} \\ &= \widehat{M}_s \end{aligned}$$

Recall

$$\mathcal{F}_\lambda = \{A \in \mathcal{F} : \ \forall t, \quad A \cap \{\lambda \le t\} \in \mathcal{F}_t\}$$

Back to original problem:

$$\widetilde{S}_T = \widehat{S}_{\tau_T} \qquad \tau_T = \int_0^T \sigma_u^2 du$$

$$P^* \sim (2) \Leftrightarrow \equiv^- \leq \int_0^T \sigma_u^2 du \leq \equiv^+$$
$$\Leftrightarrow \equiv^- \leq \tau_T \leq \equiv^+$$

It follows:

$$A = \sup_{P^* \sim (2)} \Lambda_0 E^* f(\widetilde{S}_T)$$
$$= \sup_{\equiv^- \leq \tau_T \leq \ \equiv^+} \Lambda_0 E^* f(\widehat{S}_{\tau_T})$$

Also, $\widehat{S}$ is the same for all P^*

$$\widehat{S}_0 = \widetilde{S}_0 = \frac{1}{\Lambda_0} S_0 \quad d\widehat{S}_t = \widehat{S}_t d\widehat{M}_t$$

If $f =$ convex, Jensen's inequality $\Rightarrow$

$$E[f(\widehat{S}_{\equiv^+}) \mid \widehat{\mathcal{F}}_{\tau_T}] \geq f(\widehat{S}_{\tau_T}) \quad \text{since } \tau_T \leq \equiv^+$$

so

$$A = \Lambda_0 E^+ f(\widehat{S}_{\equiv^+}) = \text{ BS}(S_0, -\log \Lambda_0, \equiv^+)$$

If $f =$ concave:

$$E[f(\widehat{S}_{\tau_T}) \mid \mathcal{F}_{\equiv^-}] \leq f(\widehat{S}_{\equiv^-})$$

$$A = \Lambda_0 E^* f(\widehat{S}_{\equiv^-}) = BS(S_0, -\log \Lambda_0, \equiv^-)$$

Integral bound + traded options?

Ask price A for payoff $f(\widetilde{S}_T) = (\widetilde{S}_T - K)^+$ under call prices

$$\Lambda_0 E^*(\widetilde{S}_T - K_i)^+ = C_0^{(i)} \quad i = 1, \ldots, p \tag{5}$$

Set

$$h_\lambda(s) = (s - K)^+ + \sum_{i=1}^{P} \lambda_i [(s - K_i)^+ - \widetilde{C}_0^{(i)}]$$

$$A \leq \inf_\lambda \sup_{P^* \sim (2)} \Lambda_0 E^* h_\lambda(\widetilde{S}_T)$$

since one can replicate $h_\lambda(\widetilde{S}_T)$ instead of $f(\widetilde{S}_T)$. Any $\lambda \Rightarrow$ sup over P^* satisfying (5):

$$A \leq \sup_{P^* \sim (2),(5)} \Lambda_0 E^*(\widetilde{S}_T - K)^+$$

Some regularity conditions: $A =$ this supremum

Time change:

$$A = \sup_{\tau} \Lambda_0 E^*(\widehat{S}_\tau - K)^+$$

where $\tau =$ stopping time:

$$\equiv^- \leq \tau \leq \equiv^+ \qquad (2)'$$

$$\Lambda_0 E^*(\widehat{S}_\tau - K_i)^+ = C_0^{(i)} \qquad (5)'$$

$\widehat{S}$ as before: $d\widehat{S}_t = \widehat{S}_t d\widehat{M}_t$

If τ' solves problem (in place of τ), set

$$\tau = \inf\{t \geq \tau' \quad \widehat{S}_t = K_i\} \wedge \equiv^+,$$

then

$$E^*[(\widehat{S}_\tau - K_i)^+ \mid \mathcal{F}_{\tau'}] = (S_{\tau'} - K_i)^+$$

and (Jensen)

$$E^*(\widehat{S}_\tau - K)^+ \geq E^*(\widehat{S}_{\tau'} - K)^+$$

Can take

$$\widehat{S}_\tau = K_i \quad \text{for some } i, \text{ or } \tau = \equiv^+$$

Simple case: all calls have some implied volatility $\equiv$:

$$BS(S_0, -\log \Lambda_0, \equiv) = \Lambda_0 E^*(\widehat{S}_{\equiv} - K_i)^+ = C_0^{(i)}$$

Can take

$$\tau' = \equiv$$

This gives optimal A.