

LECTURE 12: BID AND ASK HEDGING

1. INTRODUCTION

One of the consequences of incompleteness is that the price of derivatives is no longer unique. Various strategies for dealing with this exist, but a useful one is to consider the whole interval of such prices, and look at the upper end as a form of ask price, and the lower end as a bid price. This is a surprisingly powerful tool. Under some natural conditions, bid and ask prices have to be endpoints of such intervals. On the other hand, it turns out that one can actually hedge the bid and ask prices. We shall see that one can consider such intervals in a wide variety of contexts, including jumps, statistical uncertainty, restrictions on borrowing and short selling, and transaction costs.

In many circumstances, the supremum over all possible prices will be too high to be an ask price. In such cases, it can still be useful to use the intervals for risk management purposes. Various modifications are also possible in the form of quantile hedging and prediction intervals.

2. THE EQUIVALENCE BETWEEN INTERVALS AND BID-ASK SPREADS

Consider the one period model. What would be a natural model for the ask price at time zero for payoff η at time 1?

Definition 1. *The functional $\eta \rightarrow A(\eta)$ is a (one period) ask price if it satisfies the following:*

$$(1) \quad A(\eta_1 + \eta_2) \leq A(\eta_1) + A(\eta_2) ;$$

$$(2) \quad \text{for } \alpha \text{ scalar, } \alpha \geq 0; \quad A(\alpha\eta) = \alpha A(\eta);$$

$$(3) \quad \text{for } \eta \geq 0 : \quad A(-\eta) \leq 0;$$

and

$$(4) \quad A(-1) = -A(1).$$

Similarly, the bid price $B(\eta)$ satisfies the same assumptions, but with the inequality in (1) reversed and with (3) replaced by $B(\eta) \geq 0$.

The arguments for (1) and (3) are arbitrage based. (2) is based on the assumption that the trader is indifferent to quantity. The latter, of course, is an oversimplification. (4) asserts that there is no bid-ask spread on the bond that pays one unit of currency at time 1. This is, obviously, also an idealization.

If A is an ask price, then an associated bid price is naturally given by

$$(5) \quad B(\eta) = -A(-\eta)$$

and vice versa. Because of this connection, we shall mostly ignore bid prices in our discussion, since all results concerning these can be obtained from results on ask prices.

The relationship between bid and ask prices and sets of probabilities is now the following.

Theorem 1. *Subject to regularity conditions, A is a (one period) ask price if and only if there is a convex set \mathcal{P}^* of probability distributions so that, for all bounded η ,*

$$(6) \quad A(\eta) = A(1) \sup_{P^* \in \mathcal{P}^*} E^*(\eta).$$

Similarly, B is a bid price if and only if it can be represented on the form (6) with \inf replacing \sup .

The result is proved in Section 6. Note that the assumption that η be bounded can be weakened substantially.

This means, on the one hand, that a set \mathcal{P}^* of probability measures can give rise to bid and ask prices, and on the other hand, that bid and ask prices induce a set of risk neutral measures.

$A(1)$ is the price at time zero of payoff one yuan at time 1. If the distance between the two time points is small, this represents the short rate. Otherwise, it is the price of the zero coupon bond.

How does this play out in multiperiod setting? Assume that a payoff η , to occur at time T , is given. If $A_t(\eta)$ is the ask price for this payoff at time t , then, obviously, $A_T(\eta) = \eta$. Also, it is sensible to require, recursively, that for $s \leq t$

$$(7) \quad A_s(\eta) = A_s(A_t(\eta)).$$

In other words, the ask price at s of η is the same as the ask price at s to cover the ask price for η at t .

To set up the “short rate”, let B_t be the value of the money market bond. Suppose first that time is discrete: $0 = t_0 < t_1 < \dots < t_n = T$. We then set:

$$(8) \quad \frac{B_{t_i}}{B_{t_{i+1}}} = A_{t_i}(\text{payoff 1 unit at time } t_{i+1}) \text{ with } B_0 = 1$$

In the general case, one can proceed by taking limits. Note that one can also take the t_i s to be a subset of all possible time points. In this case, B_t has a different interpretation than the short rate – it is a sequence of bonds instead. We then get (in a slightly nonrigorous formulation):

Theorem 2. *Subject to regularity conditions, the following statements are equivalent:*

- (i) A_t behaves like a one period ask price for any subsequent payoff, and (7) is valid.
- (ii) There is a convex set \mathcal{P}^* of probability distributions so that, for all bounded η with payoff at time T ,

$$(9) \quad A_t(\eta) = \text{ess sup}_{P^* \in \mathcal{P}^*} E^*(B_t B_T^{-1} \eta | \mathcal{F}_t).$$

“ess sup” means the “essential” supremum over all P^* in \mathcal{P}^* . Measure theory aficionados should consult Mykland (2000, 2003), especially the appendix and the references therein, for details. In the discrete case, “ess sup” is the same as “sup”.

The proof of the result is as painful in the continuous time general space case as it is obvious in the discrete time and space case.

3. HEDGING OF ASK PRICES

If the set \mathcal{P}^* of risk neutral distributions is big enough, then one can hedge the resulting ask price. The main difference with what we have done before is that we now encounter super-replications. The trading strategy is self financing, but there may be dividends.

Consider traded securities $S_t^{(1)}, \dots, S_t^{(p)}$, paying no dividends. The risk free interest rate is r_t , and $B_t = \exp\{\int_0^t r_u du\}$ is the value at time t of one yuan deposited in the money market at time 0. \mathcal{P} is a set of probability distributions.

We find ourselves in the following situation. We stand at time $t = 0$, and we have to make a payoff η at a (non random or stopping) time τ . η is \mathcal{F}_τ -measurable. We do not know what the probability distribution for this system is, but we know that it is an element in \mathcal{P} . We are looking for hedging strategies in $S_t^{(1)}, \dots, S_t^{(p)}, B_t$ that will super-replicate the payoff with probability one.

Definition 2. A property will be said to hold \mathcal{P} - a.s. if it holds P - a.s. for all $P \in \mathcal{P}$. For any process X_t , $\tilde{X}_t = B_t^{-1} X_t$, and vice versa.

A process $V_t, 0 \leq t \leq T$, is said to be a super-replication of payoff η provided

(i) one can cover one's obligations:

$$(10) \quad V_\tau \geq \eta$$

\mathcal{P} - a.s.; and

(ii) for all $P \in \mathcal{P}$, there are processes H_t and D_t , so that, for all $t, 0 \leq t \leq T$,

$$(11) \quad V_t = H_t - D_t, \quad 0 \leq t \leq T,$$

where \tilde{D}_t is a nondecreasing process, and where H_t is self financing in the traded securities $B_t, S_t^{(1)}, \dots, S_t^{(p)}$.

What should the price be for a promise of a payoff η ? A natural hedging based approach is as follows, cf. Cvitanic and Karatzas (1992, 1993), El Karoui and Quenez (1995), and Kramkov (1996).

Definition 3. The hedge based ask price at time 0 for a payoff η to be made at a time τ is

$$(12) \quad A_0 = \inf\{V_0 : (V_t) \text{ is a super-replication of the payoff}\}.$$

Similarly, the hedge based bid price can be defined as the supremum over all sub-replications of the payoff, in the obvious sense.

To give the general form of the ask price A , we consider an appropriate set \mathcal{P}^* of "risk neutral" probability distributions P^* .

Definition 4. For given \mathcal{P} , numeraire B_t , and other traded securities $S_t^{(1)}, \dots, S_t^{(p)}$, \mathcal{P}^* is the set of all probability distributions P^* so that

- (i) the $\tilde{S}^{(i)} = S^{(i)}/B_t$ are martingales; and
- (ii) if $P(A) = 0$ for all $P \in \mathcal{P}$, then $P^*(A) = 0$.

The result is then as follows:

Theorem 3. Let \mathcal{P} , numeraire B_t , and other traded securities $S_t^{(1)}, \dots, S_t^{(p)}$ be given, and let η be a payoff at time τ . Let \mathcal{P}^* be formed as in the definition above. Subject to regularity conditions, A_0 given by (12) equals

$$(13) \quad A_0(\eta) = \sup_{P^* \in \mathcal{P}^*} E^*(B_\tau^{-1} \eta)$$

Equation (9) applies similarly.

A related result states that a process V_t can be represented on the form (11) if and only if it is a supermartingale for all $P^* \in \mathcal{P}^*$. Versions of these result are proved in Kramkov (1996) and Mykland (2000). Supermartingales are defined as follows.

Definition 5. Let (\mathcal{F}_t) be a filtration, and let Q be a probability. Then V_t is a supermartingale (with respect to (\mathcal{F}_t) and Q) if, for all $s, t, s \leq t$,

$$(14) \quad V_s \geq E_Q(V_t | \mathcal{F}_s).$$

4. CASE STUDY: THE AMERICAN OPTION

This case follows the above description, but with the additional bonus that the bid and ask prices coincide. Suppose that the interest rate r is constant and that the stock price follows a geometric Brownian motion

$$(15) \quad dS_t = \mu S_t dt + \sigma S_t dW_t.$$

The American option has payoff $\eta = f(S_\tau)$ at time τ , where τ is at the discretion of the owner. \mathcal{P} is thus the set of all probability distributions where S_t follows (15), and τ can have any distribution concentrated on $[0, T]$. The price (13) is as given in Duffie (1996), Section 8E-8F, to which we refer for further details. Note that \mathcal{P}^* is different for the seller and the buyer (the buyer can control her exercise time). Hence the bid price for buying is the infimum over just one probability, and it can be shown to coincide with the ask price.

5. CASE STUDY: UNCERTAIN VOLATILITY AND INTEREST

Suppose that r_t is random and that

$$(16) \quad dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.$$

We shall now assume that we are not willing to write down a model for r_t or σ_t , but instead assume that they belong to some set.

We shall in the following assume that this set has the form

$$(17) \quad R^+ \geq \int_0^T r_u du \geq R^- \text{ and } \Xi^+ \geq \int_0^T \sigma_u^2 du \geq \Xi^-.$$

This follows Mykland (2000). Another development, where r is constant and σ_t is bounded for each t , can be found in Avellaneda, Levy, and Paras (1995) and Lyons (1995).

For a European option with payoff $f(S_T)$, the ask price becomes:

$$(18) \quad A_0 = \sup E^* \exp\left\{-\int_0^T r_u du\right\} f(S_T),$$

where the supremum is over all risk neutral measures under which the set (17) gets probability 1.

For simplicity, consider first the case where $R^- = R^+ = R$. Since $S_T = e^R \tilde{S}_T$,

$$(19) \quad A_0 = \sup E^* e^{-R} f(e^R \tilde{S}_T).$$

One can now do a time change so that $\tilde{S}_t = \hat{S}(\int_0^t \sigma_u^2 du)$, where

$$(20) \quad d\hat{S}_t = \hat{S}_t d\hat{W}_t \text{ with } \hat{S}_0 = S_0.$$

Our price then reduces to

$$(21) \quad A_0 = \sup_{\Xi^+ \geq \tau \geq \Xi^-} E^* e^{-R} f(e^R \hat{S}_\tau),$$

which thus has the form of an American option. In this instance, however, the problem is a little simpler, since we have eliminated interest in our stopping criterion. In fact, if f is (non-strictly) convex (this is the case for call and put options), Jensen's inequality yields that

$$(22) \quad A_0 = E^* e^{-R} f(e^R \hat{S}_{\Xi^+}),$$

which is the Black-Scholes price for cumulative volatility Ξ^+ and cumulative interest R .

6. PROOF OF THEOREM 1

The “if” case is obvious. For the “only if” case, assume without loss of generality that $A(1) = 1$. Also, we shall only show this in the discrete case, hence assume that there are k possible outcomes, and that the payoff is η_i in the case of outcome number i ($i = 1, \dots, k$). A can then be seen as a function $R^d \rightarrow R$. One now goes to Theorem 12.1 (p. 102) in Rockafellar (1970), which asserts that A , being convex by assumption, is the pointwise supremum over all affine functions H that satisfy $A \geq H$. In other words,

$$(23) \quad A(\eta) = \sup_{H \in \mathcal{H}} H(\eta)$$

where $\mathcal{H} = \{H \text{ affine} : \forall \eta : H(\eta) \leq A(\eta)\}$. Our task is to show that \mathcal{H} can be replaced in (23) by \mathcal{P}^* , which we shall take to be the set of $H \in \mathcal{H}$ that are linear and satisfy $H(1) = 1$ and $H(\eta) \geq 0$ whenever $\eta \geq 0$. This makes \mathcal{P}^* a set of expectation operators, and hence the result will have been proved.

To show that the functions H can be taken to be linear, note that any affine function H can be written $H = H_l + c$, where c is a constant. If $H \in \mathcal{H}$, then by (2), $0 = A(0) \geq H(0) = c$, and hence $H_l \geq H$. On the other hand, for $\alpha > 0$, $\alpha A(\eta) = A(\alpha\eta) \geq H(\eta) = \alpha H_l(\eta) + c$, and so

$$(24) \quad A(\eta) \geq H_l(\eta) + \frac{c}{\alpha} \rightarrow H_l(\eta) \text{ as } \alpha \rightarrow \infty.$$

For a linear function $H \in \mathcal{H}$, we get that

$$(25) \quad H(\eta) = -H(-\eta) \geq -A(-\eta).$$

Hence, if $\eta \geq 0$, $H(\eta) \geq 0$ by (3). Also, (25) yields that $A(1) \geq H(1) = -H(-1) \geq -A(-1) = A(1)$ from (4). The result follows. ■

7. REFERENCES

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