STAT 391/FINMATH 346

Midterm, Wednesday, February 2, 2005

Conciseness, elegance, and legibility may be a factor in awarding credit.
All solutions should be as explicit as possible.

1. The old CBOE cap. We assume in the following that under the true probability distribution P, the stock price S is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t. \tag{1}$$

Also suppose that the short rate r is constant under P.

We shall be concerned with the option which pays $\eta = (S_\tau - K)^+$ dollars at time τ, where

$$\tau = T \wedge \min\{t \geq 0 : S_t = X\}$$

Here, $X > \max(S_0, K)$.

(a) Explain why this payoff can be hedged (keep in mind that the time τ is random).

(b) Let P^* be the risk neutral measure, and let Q be the measure under which $B_t = (\log S_t - \log S_0)/\sigma$ is a standard Brownian motion. Give dP^*/dQ.

(c) Derive the value $C(S_0, T)$ of the option at time $t = 0$ as an expected value under Q.

(d) Calculate, as explicitly as you can, the value $C(S_0, T)$.

(e) Derive the hedging strategy for the option in terms of the function $C(s, u)$.

[Background: This security used to be traded on the CBOE, but was found to be hard to hedge in practice.]

2. A perpetual double barrier option. We assume that P is as in Problem 1 above. P^* is the equivalent martingale measure. We shall be concerned with the option which pays η_τ dollars at time τ, where

$$\eta_\tau = \begin{cases} (S_\tau - K)^+ \text{ if } S_t \text{ never crosses or touches the barrier } X_1 \text{ and } X_2 \text{ between times 0 and } \tau \\ 0 \text{ otherwise} \end{cases}$$

Here, $X_1 < \min(S_0, K)$ and $X_2 > \max(S_0, K)$. τ is determined by the holder of the option, and there is no upper limit to hold long the option can be held.

(a) Let λ be the time

$$\lambda = \min\{t \geq 0 : S_t = X_1 \text{ or } X_2\}.$$

Condsider two stopping times τ_1 and τ_2, where $0 \leq \tau_1 \leq \tau_2 < \lambda$. Which of the following is true:
(i) $E^* e^{-r\tau_1}\eta_{\tau_1} \leq E^* e^{-r\tau_2}\eta_{\tau_2}$
(ii) $E^* e^{-r\tau_1}\eta_{\tau_1} \geq E^* e^{-r\tau_2}\eta_{\tau_2}$
(iii) Either inequality can occur, depending on the specific form of τ_1 and τ_2.
Justify your answer.

(b) Determine whether an optimal exercise strategy exists. If it does, provide it. If it does not, explain why.

(c) Relate the value V_0 at time 0 of this security to the value V_0' of the security which pays $(S_\lambda - K)^+$ at time λ.

(d) If $r = 0$, calculate V_0. [If you cannot do that, calculate V_0' instead.]

3. A binary incompleteness. We assume in the following that under the true probability distribution P, the stock price S is given by

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t. \tag{2}$$

μ_t and σ_t are random, but $\Xi = \int_0^T \sigma_t^2 dt$ can take only two possible values Ξ^- and Ξ^+. We suppose also that for known nonrandom times T' and T'', $0 < T' < T'' < T$, $\int_0^{T'} \sigma_t^2 dt = \Xi^-$ and $\int_{T'}^{T''} \sigma_t^2 = 0$. Also, we suppose that the short rate $r = 0$ under P.

Finally, there is a strike price X so that the call option which pays $(S_T - X)^+$ dollars at time T is traded in the market. The value of this option at time t is V_t^X. We suppose that for $0 \le t \le T', d < V, S >_t /dt = f(S_t, t)$ for a known function $f(s, t)$.

We shall be interested in finding the price V_t^K that we need to hedge a call option with payoff (at time T) $(S_T - K)^+$, where $K \neq X$.

(a) Express the price $V_{T'}^K$ in terms of $S_{T'}$ and $V_{T'}^X$

(b) Find V_t^K for all t (as explicitly as you can), and describe the hedging strategy for V_t^K in S_t and V_t^X.

(c) [For bonus points only] Let P^* be the risk neutral measure, and let $(\mathcal{F}_t)$ be the filtration induced by S and V^X. Find $P^*(\Xi = \Xi^+ | \mathcal{F}_t)$ for all t, in terms of S and V, as explicitly as you can.