1. The old CBOE cap. We assume in the following that under the true probability distribution

\[ P, \]

the stock price \( S \) is given by

\[ dS_t = \mu S_t dt + \sigma S_t dW_t. \quad (1) \]

Also suppose that the short rate \( r \) is constant under \( P \).

We shall be concerned with the option which pays \( \eta = (S_\tau - K)^+ \) dollars at time \( \tau \), where

\[ \tau = T \wedge \min\{t \geq 0 : S_t = X\} \]

Here, \( X > \max(S_0, K) \).

(a) Explain why this payoff can be hedged (keep in mind that the time \( \tau \) is random).

(b) Let \( P^* \) be the risk neutral measure, and let \( Q \) be the measure under which \( B_t = (\log S_t - \log S_0)/\sigma \) is a standard Brownian motion. Give \( dP^*/dQ \).

(c) Derive the value \( C(S_0, T) \) of the option at time \( t = 0 \) as an expected value under \( Q \).

(d) Calculate, as explicitly as you can, the value \( C(S_0, T) \).

(e) Derive the hedging strategy for the option in terms of the function \( C(s, u) \).

[Background: This security used to be traded on the CBOE, but was found to be hard to hedge in practice.]

2. A perpetual double barrier option. We assume that \( P \) is as in Problem 1 above. \( P^* \) is the equivalent martingale measure. We shall be concerned with the option which pays \( \eta_\tau \) dollars at time \( \tau \), where

\[ \eta_\tau = \begin{cases} 
(S_\tau - K)^+ & \text{if } S_\tau \text{ never crosses or touches the barrier } X_1 \text{ and } X_2 \text{ between times 0 and } \tau \\
0 & \text{otherwise}
\end{cases} \]

Here, \( X_1 < \min(S_0, K) \) and \( X_2 > \max(S_0, K) \). \( \tau \) is determined by the holder of the option, and there is no upper limit to how long the option can be held.

(a) Let \( \lambda \) be the time

\[ \lambda = \min\{t \geq 0 : S_t = X_1 \text{ or } X_2\}. \]

Consider two stopping times \( \tau_1 \) and \( \tau_2 \), where \( 0 \leq \tau_1 \leq \tau_2 < \lambda \). Which of the following is true:

(i) \( E^* e^{-r\tau_1} \eta_{\tau_1} \leq E^* e^{-r\tau_2} \eta_{\tau_2} \)

(ii) \( E^* e^{-r\tau_1} \eta_{\tau_1} \geq E^* e^{-r\tau_2} \eta_{\tau_2} \)

(iii) Either inequality can occur, depending on the specific form of \( \tau_1 \) and \( \tau_2 \).

Justify your answer.

(b) Determine whether an optimal exercise strategy exists. If it does, provide it. If it does not, explain why.

(c) Relate the value \( V_0 \) at time 0 of this security to the value \( V'_0 \) of the security which pays \( (S_\lambda - K)^+ \) at time \( \lambda \).

(d) If \( r = 0 \), calculate \( V_0 \). [If you cannot do that, calculate \( V'_0 \) instead.]
3. A binary incompleteness. We assume in the following that under the true probability distribution $P$, the stock price $S$ is given by
\[
dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.
\] (2)
where $\mu_t$ and $\sigma_t$ are random, but $\Xi = \int_0^T \sigma_t^2 dt$ can take only two possible values $\Xi^-$ and $\Xi^+$. We suppose also that for known nonrandom times $T'$ and $T''$, $0 < T' < T'' < T$, $\int_0^{T'} \sigma_t^2 dt = \Xi^-$ and $\int_T^{T''} \sigma_t^2 = 0$. Also, we suppose that the short rate $r = 0$ under $P$.

Finally, there is a strike price $X$ so that the call option which pays $(S_T - X)^+$ dollars at time $T$ is traded in the market. The value of this option at time $t$ is $V_t^X$. We suppose that for $0 \leq t \leq T'$, $d < V; S >_t /dt = f(S_t, t)$ for a known function $f(s, t)$.

We shall be interested in finding the price $V_t^K$ that we need to hedge a call option with payoff (at time $T$) $(S_T - K)^+$, where $K \neq X$.

(a) Express the price $V_t^K$ in terms of $S_{T'}$ and $V_t^X$.

(b) Find $V_t^K$ for all $t$ (as explicitly as you can), and describe the hedging strategy for $V_t^K$ in $S_t$ and $V_t^X$.

(c) [For bonus points only] Let $P^*$ be the risk neutral measure, and let $(\mathcal{F}_t)$ be the filtration induced by $S$ and $V^X$. Find $P^*(\Xi = \Xi^+ | \mathcal{F}_t)$ for all $t$, in terms of $S$ and $V$, as explicitly as you can.