CONTINUOUS TIME PRICING AND TRADING:

A REVIEW, WITH SOME EXTRA PIECES

THE SOURCE OF A PRICE IS ALWAYS A TRADING STRATEGY

- SPECIAL CASES WHERE TRADING STRATEGY IS INDEPENDENT OF PROBABILITY MEASURE

- COMPLETENESS, RISK NEUTRAL PRICING
  including quantile hedging (in the lect 7 case)

- SUPERREPLICATION
  American options
  unknown probability measure
  known probability measure, incomplete market

- MEAN-VARIANCE, UTILITY BASED PRICING
  such as the paper by Schweizer
  unknown probability measure
  known probability measure, incomplete market
THE SIMPLEST CASE

Problem: A contract pays the owner 1 share of stock $S$ at the first time $t, 0 \leq t \leq T$ that the share price $S_t$ exceeds $X$. If the share price does not exceed $X$ at any time $t = 0, ..., T$, the contract pays one share of stock $S$ at time $T$. Find the price of this contract.

Solution: To satisfy this contract, you need to buy one share of stock at time zero.

THIS IS A TRADING STRATEGY, BUT DOES NOT REQUIRE STOCHASTIC CALCULUS

IF SUCH A STRATEGY IS AVAILABLE, USE IT!

OTHER EXAMPLES

• prices of forward contracts
• put call parity for European options
IN MOST CASES:

REDUCE PROBLEM BY NUMERAIRE INVARIANCE

\[ \tilde{S}_t = \text{discounted price} \quad \tilde{\eta} = \text{discounted payoff at } T \]

\( \tilde{\eta} \) can be exactly financed if and only if

\[ \tilde{\eta} = \tilde{c} + \int_0^T \theta_t d\tilde{S}_t \] (*)

(Lect 9 last quarter, p. 9-10)

\( \tilde{c} \) is initial (discounted) price

\( \theta_t \) is the “delta” (independent of numeraire)

• This is a “self financing strategy” (SFS)

• Same principle for multiple securities

• This does not depend on “risk free” or “actual” measure

IF YOU FIND SUCH A STRATEGY W/OUT GOING THROUGH THE USUAL MACHINERY:
USE IT!
CLASSICAL EXAMPLE: THE VOLATILITY SWAP

\[ d\tilde{S}_t = \tilde{S}_t \mu_t dt + \tilde{S}_t \sigma_t dW_t \]

Ito’s formula: \( \log \tilde{S}_T = \log \tilde{S}_0 + \int_0^T \frac{1}{\tilde{S}_t} d\tilde{S}_t - \frac{1}{2} \int_0^T \sigma_t^2 dt \)

Or: \( \int_0^T \sigma_t^2 dt = 2 \left( -\log \tilde{S}_T + \log \tilde{S}_0 + \int_0^T \frac{1}{\tilde{S}_t} d\tilde{S}_t \right) \)

Read directly from this that discounted payoff \( \int_0^T \sigma_t^2 dt \) can be replicated by:

- initial capital: \( 2 \log \tilde{S}_0 \)
- + by owning an option with payoff \( -2 \log \tilde{S}_T \)
- Dynamic hedge: \( \theta_t = \frac{2}{\tilde{S}_t} \)

Note: we have not told you which probability distribution, \( P \) or \( P^* \), we are using
VOLATILITY SWAP, CONTINUED

If you wish to replicate actual (not discounted payoff) \( \int_0^T \sigma_t^2 dt \):

Suppose discounting by zero coupon bond \( \Lambda_t, \Lambda_T = 1 \)
- The discounted payoff is \( \frac{1}{\Lambda_0} \int_0^T \sigma_t^2 dt \)
- Therefore: same strategy as on previous page, multiplied by \( \frac{1}{\Lambda_0} \)

WHICH MEANS REPLICATION BY:
- initial capital: \( \frac{2}{\Lambda_0} \log S_0 \)
- + by owning an option with payoff \(- \frac{2}{\Lambda_0} \log \tilde{S}_T \)
- Dynamic hedge: \( \theta_t = \frac{2}{\Lambda_0 S_t} \)

IMPORTANT:

THIS IS A HEDGE FOR THE CUMULATIVE VOLATILITY OF THE DISCOUNTED SECURITY

(This is the same as the cum. vol. for the original security if \( r \) is constant)
THE "USUAL MACHINERY"

IN MORE COMPLEX CASES, CANNOT READ HEDGE DIRECTLY

SIMPLEST APPROACH: COMPLETENESS FROM GEOMETRIC BROWNIAN MOTION

system: \(dS_t = \mu_t S_t dt + \sigma S_t dW_t\) and \(r = \) constant
payoff: \(\eta = \) function of the path of \(S\)

IN THIS CASE, THE ALGORITHM IS...

- Define \(P^*\) to be such that \(dS_t = rS_t dt + \sigma S_t dW_t\)
- Compute \(\tilde{C}_t = E^*(\tilde{\eta} \mid \mathcal{F}_t)\)
- The delta is \(\theta_t = \frac{d[\tilde{C}, \tilde{S}]_t}{d[S, S]_t}\)

This works (gives SFS) because of the martingale representation theorem (p. 11-12 in Lect 9 of last quarter)

YOU NEED TO KNOW HOW TO DO THIS

FOR MORE EXERCISE, COMPUTE THE ANALYTIC EXPRESSIONS GIVEN IN HULL’S BOOK
A MORE COMPLEX COMPLETE (?) CASE:

THE HESTON MODEL

\[
dS_t = \mu_t S_t dt + v_t^{1/2} S_t dW_t
\]

\[
dv_t = a(b - v_t) dt + cv_t^{1/2} dB_t
\]

with \(d[W, B]_t = \rho dt\) and constant interest rate \(r\)

System is generated by two Brownian motions. Two possibilities:

- if only \(S\) is traded: market is incomplete, need to use methods to this case
- if one derivative is traded, may be able to complete market with this derivative (need as many securities as you have Brownian motions)

But a problem is as follows: under the risk neutral measure

\[
dS_t = rS_t dt + v_t^{1/2} S_t dW^*_t
\]

\[
dv_t = ??? dt + cv_t^{1/2} dB^*_t
\]

The market may not be complete under \(P^*\)
BACK TO THE DRAWING BOARD...

Suppose the model is valid under $P^*$

$$dS_t = rS_t dt + v_t^{1/2} S_t dW^*_t$$
$$dv_t = a(b - v_t)dt + cv_t^{1/2} dB_t$$

Suppose for simplicity that $\rho = 0$ (otherwise numerical solution only). European call payoff $\eta^K = (S_T - K)^+$. If $\tilde{B}(\tilde{S}_t, \sigma^2(T - t))$ is Black-Scholes price (for constant $\sigma^2$), then, since $\rho = 0$:

$$E^*(\tilde{\eta}^K | (v_u)_{0 \leq u \leq T}, (S_u)_{0 \leq u \leq t}) = B(\tilde{S}_t, \int_t^T v_u du)$$

and so the discounted price for payoff $\eta^K$ is

$$\tilde{C}_t^K = E^*(\tilde{\eta}^K | \mathcal{F}_t) = E^*(B(\tilde{S}_t, \int_t^T v_u du) | \mathcal{F}_t)$$
$$= f^K(\tilde{S}_t, v_t, T - t)$$

which can be calculated if one knows $a, b, c$
HOW TO HEDGE IN THIS MODEL

Since the system is generated by two Brownian motions under the risk neutral measure: need two securities to hedge, say $S_t$, and $C^X_t$ for one strike price $X$

If $f^K_v$ means $f^K_v(\tilde{S}_t, v_t, T - t)$, etc, and since $d[\tilde{S}, v]_t = 0$

\[
\begin{align*}
d\tilde{C}^K_t &= f^K_S d\tilde{S}_t + \frac{1}{2} f^K_{SS} d[\tilde{S}, \tilde{S}]_t + f^K_v dv_t + \frac{1}{2} f^K_{vv} d[v, v]_t - f^K_t dt
\end{align*}
\]

and the same for $C^X$, so that

\[
\begin{align*}
d\tilde{C}^K_t &= f^K_S d\tilde{S}_t + f^K_v dv_t + dt\text{-terms} \\
d\tilde{C}^X_t &= f^X_S d\tilde{S}_t + f^X_v dv_t + dt\text{-terms}
\end{align*}
\]

Express the $dv_t$-term by

\[
\begin{align*}
dv_t &= \frac{1}{f^X_v} d\tilde{C}^X_t - \frac{f^X_S}{f^X_v} d\tilde{S}_t + dt\text{-terms}
\end{align*}
\]

and get

\[
\begin{align*}
d\tilde{C}^K_t &= f^K_S d\tilde{S}_t + f^K_v \left( \frac{1}{f^X_v} d\tilde{C}^X_t - \frac{f^X_S}{f^X_v} d\tilde{S}_t \right) + dt\text{-terms}
\end{align*}
\]
Rearrange:

\[ d\tilde{C}_t^K = \left( f^K_S - \frac{f^K_v f^K_X}{f^K_v} \right) d\tilde{S}_t + \frac{f^K_v}{f^K_X} d\tilde{C}_t^X + dt\text{-terms} \]

However, \( \tilde{C}_t^K, \tilde{S}_t \) and \( \tilde{C}_t^X \) are all (local) martingales, so

\[ dt\text{-terms} = 0 \]

Final self financing strategy:

\[ d\tilde{C}_t^K = \left( f^K_S - \frac{f^K_v f^K_X}{f^K_v} \right) d\tilde{S}_t + \frac{f^K_v}{f^K_X} d\tilde{C}_t^X \]

in other words: to hedge payoff \( \eta^K \), hold

- \( f^K_S - \frac{f^K_v f^K_X}{f^K_v} \) units of stock \( S_t \)
- \( \frac{f^K_v}{f^K_X} \) units of the option with payoff \( \eta^X \)

Initial starting capital: \( f(\tilde{S}_0, v_0, T) \) dollars

THE ONLY THING THAT REMAINS IS TO CALCULATE THE FUNCTION \( f \)
ANOTHER CASE OF HEDGING
IN ADDITIONAL SECURITIES

The superreplication in HW 6

THE THREE INVARIANCES

SELF FINANCING STRATEGIES ARE INVARIANT UNDER:

- Change of numeraire
- Change of measure (so long as absolutely continuous)
- Change of time
CHANGE OF TIME

Function \( f : [0, T] \rightarrow [0, T'] \) is a time change if

- \( f \) is increasing
- \( f(0) = 0 \) and \( f(T') = T' \)
- \( f^{-1}(t) \) is a stopping time, for each \( t \)

Securities on original time scale \( \tilde{S}_t, \tilde{C}_t \) are connected by

\[
\tilde{C}_t = \tilde{C}_0 + \int_0^t \theta_u d\tilde{S}_u
\]

Securities on new time scale

\[
\tilde{S}_t^{\text{new}} = \tilde{S}_{f(t)} \quad \text{and} \quad \tilde{C}_t^{\text{new}} = \tilde{C}_{f(t)}
\]

satisfy

\[
\tilde{C}_t^{\text{new}} = \tilde{C}_0 + \int_0^t \theta_u^{\text{new}} d\tilde{S}_u^{\text{new}}
\]

where

\[
\theta_t^{\text{new}} = \theta_{f(t)}
\]
This is because the stochastic integral is a limit of sums:

Set $u_i = f(v_i)$:

\[
\tilde{C}_{t_{\text{new}}} - \tilde{C}_0 = \int_0^{f(t)} \theta_u d\tilde{S}_u \\
\approx \sum_{u_i < f(t)} \theta_{u_i} (\tilde{S}_{u_{i+1}} - \tilde{S}_{u_i}) \\
= \sum_{v_i < t} \theta_{f(v_i)} (\tilde{S}_{f(v_{i+1})} - \tilde{S}_{f(v_i)}) \\
= \sum_{v_i < t} \theta_{v_i}^{\text{new}} (\tilde{S}_{v_{i+1}}^{\text{new}} - S_{v_i}^{\text{new}}) \\
\approx \int_0^t \theta_u^{\text{new}} d\tilde{S}_u
\]
IMPORTANT CONSEQUENCE OF INVARIANCE TO TIME CHANGE

Suppose that the discounted payoff $\tilde{\eta}$ satisfies:

$$\tilde{\eta} = g(\tilde{S})$$

where $g$ is invariant to time:

if $s^{\text{new}}_t = s_{f(t)}$ for any deterministic time change, then $g(s^{\text{new}}) = g(s)$

Example: European payoffs, barrier and lookback payoffs written on the discounted process, or future

THEN: if you know that $\int_0^T \sigma_t^2 dt = \Xi$, you can price option as if $\sigma^2_t = \Xi/T = \text{constant}$

Technically: time change on the form:

$$f^{(-1)}(t) = \int_0^t \sigma_u^2 du$$

The time changed security is

$$d\tilde{S}^{\text{new}}_t = \sqrt{\frac{\Xi}{T}} \tilde{S}^{\text{new}}_t dW^{\text{new}}$$

(see p. 4-6 in Lect 5 (part 2))
CONVEXITY AND OPTIMAL STOPPING

JENSEN’S INEQUALITY

$g$ is a convex function, $M_t$ is a martingale, $\tau_1$, $\tau_2$ are stopping times, with $0 \leq \tau_1 \leq \tau_2 \leq T$

$$E(g(M_{\tau_2}) \mid \mathcal{F}_{\tau_1}) \geq g(M_{\tau_1})$$

SEVERAL CONSEQUENCES, SUCH AS

APPLICATION 1: If the interest rate is zero, it is never optimal to exercise any convex payoff early. Application to superhedging in Lecture 5.

APPLICATION 2: The American option inequality (Lect 5 in Fall (p. 14))

Assume also $g(s) \geq 0$, $g(0) = 0$, and that $\exp\{-\int_0^t ru\,du\}S_t$ is a martingale. Then

$$\exp\{-\int_0^t ru\,du\}g(S_t)$$

is a submartingale. Then

In particular, it is never optimal to exercise an American call early (when there is no dividend)