CONTINUOUS TIME PRICING AND TRADING:

A REVIEW, WITH SOME EXTRA PIECES

THE SOURCE OF A PRICE IS ALWAYS A TRADING STRATEGY

- SPECIAL CASES WHERE TRADING STRATEGY IS INDEPENDENT OF PROBABILITY MEASURE

- COMPLETENESS, RISK NEUTRAL PRICING
  including quantile hedging (in the lect 7 case)

- SUPERREPLICATION
  American options
  unknown probability measure
  known probability measure, incomplete market

- MEAN-VARIANCE, UTILITY BASED PRICING
  such as the paper by Schweizer
  unknown probability measure
  known probability measure, incomplete market

THE SIMPLEST CASE

Problem: A contract pays the owner 1 share of stock S at the first time $t, 0 \leq t \leq T$ that the share price S_t exceeds \$ X. If the share price does not exceed X at any time $t = 0, ..., T$, the contract pays one share of stock S at time T. Find the price of this contract.

Solution: To satisfy this contract, you need to buy one share of stock at time zero

THIS IS A TRADING STRATEGY, BUT DOES NOT REQUIRE STOCHASTIC CALCULUS

IF SUCH A STRATEGY IS AVAILABLE, USE IT!

OTHER EXAMPLES

- prices of forward contracts
- put call parity for European options

IN MOST CASES:

REDUCE PROBLEM BY NUMERAIRE INVARIANCE

$\widetilde{S}_t$ = discounted price $\qquad$ $\widetilde{\eta}$ = discounted payoff at T

$\widetilde{\eta}$ can be exactly financed if and only if

$$\widetilde{\eta} = \widetilde{c} + \int_0^T \theta_t d\widetilde{S}_t \qquad (*)$$

(Lect 9 last quarter, p. 9-10)

$\widetilde{c}$ is initial (discounted) price

θ_t is the "delta" (independent of numeraire)

• This is a "self financing strategy" (SFS)

• Same principle for multiple securities

• This does not depend on "risk free" or "actual" measure

IF YOU FIND SUCH A STRATEGY W/OUT

GOING THROUGH THE USUAL MACHINERY:

USE IT!

CLASSICAL EXAMPLE: THE VOLATILITY SWAP

$$d\widetilde{S}_t = \widetilde{S}_t \mu_t dt + \widetilde{S}_t \sigma_t dW_t$$

Ito's formula: $\log \widetilde{S}_T = \log \widetilde{S}_0 + \int_0^T \frac{1}{\widetilde{S}_t} d\widetilde{S}_t - \frac{1}{2} \int_0^T \sigma_t^2 dt$

Or: $\int_0^T \sigma_t^2 dt = 2 \left(-\log \widetilde{S}_T + \log \widetilde{S}_0 + \int_0^T \frac{1}{\widetilde{S}_t} d\widetilde{S}_t \right)$

Read directly from this that discounted payoff $\int_0^T \sigma_t^2 dt$ can be replicated by:

- initial capital: $2 \log \widetilde{S}_0$
- $+$ by owning an option with payoff $-2 \log \widetilde{S}_T$
- Dynamic hedge: $\theta_t = \frac{2}{\widetilde{S}_t}$

Note: we have not told you which probability distribution, P or P^*, we are using

VOLATILITY SWAP, CONTINUED

If you wish to replicate actual (not discounted payoff) $\int_0^T \sigma_t^2 dt$:

Suppose discounting by zero coupon bond Λ_t, $\Lambda_T = 1$

- The discounted payoff is $\frac{1}{\Lambda_0} \int_0^T \sigma_t^2 dt$
- Therefore: same strategy as on previous page, multiplied by $\frac{1}{\Lambda_0}$

WHICH MEANS REPLICATION BY:

- initial capital: $\frac{2}{\Lambda_0} \log \widetilde{S}_0$
- \+ by owning an option with payoff $-\frac{2}{\Lambda_0} \log \widetilde{S}_T$
- Dynamic hedge: $\theta_t = \frac{2}{\Lambda_0 \widetilde{S}_t}$

IMPORTANT:

THIS IS A HEDGE FOR THE CUMULATIVE VOLATILITY OF THE DISCOUNTED SECURITY

(This is the same as the cum. vol. for the original security if r is constant)

THE "USUAL MACHINERY"

IN MORE COMPLEX CASES, CANNOT READ HEDGE DIRECTLY

SIMPLEST APPROACH: COMPLETENESS FROM GEOMETRIC BROWNIAN MOTION

$$\text{system: } dS_t = \mu_t S_t dt + \sigma S_t dW_t \text{ and } r = \text{ constant}$$
$$\text{payoff: } \eta = \text{ function of the path of } S$$

IN THIS CASE, THE ALGORITHM IS...

- Define P^* to be such that $dS_t = rS_t dt + \sigma S_t dW_t$
- Compute $\widetilde{C}_t = E^*(\widetilde{\eta} \mid \mathcal{F}_t)$
- The delta is $\theta_t = \frac{d[\widetilde{C},\widetilde{S}]_t}{d[\widetilde{S},\widetilde{S}]_t}$

This works (gives SFS) because of the martingale representation theorem (p. 11-12 in Lect 9 of last quarter)

YOU NEED TO KNOW HOW TO DO THIS

FOR MORE EXERCISE, COMPUTE THE ANALYTIC EXPRESSIONS GIVEN IN HULL'S BOOK

A MORE COMPLEX COMPLETE (?) CASE:

THE HESTON MODEL

$$dS_t = \mu_t S_t dt + v_t^{1/2} S_t dW_t$$
$$dv_t = a(b - v_t)dt + c v_t^{1/2} dB_t$$

with $d[W, B]_t = \rho dt$ and constant interest rate r

System is generated by two Brownian motions. Two possibilities:

• if only S is traded: market is incomplete, need to use methods to this case

• if one derivative is traded, may be able to complete market with this derivative (need as many securities as you have Brownian motions)

But a problem is as follows: under the risk neutral measure

$$dS_t = r S_t dt + v_t^{1/2} S_t dW_t^*$$
$$dv_t = ???dt + c v_t^{1/2} dB_t^*$$

The market may not be complete under P^*

BACK TO THE DRAWING BOARD...

Suppose the model is valid under P^*

$$dS_t = rS_t dt + v_t^{1/2} S_t dW_t^*$$
$$dv_t = a(b - v_t)dt + cv_t^{1/2} dB_t$$

Suppose for simplicity that $\rho = 0$ (otherwise numerical solution only). European call payoff $\eta^K = (S_T - K)^+$.

If $\widetilde{B}(\widetilde{S}_t, \sigma^2(T-t))$ is Black-Scholes price (for constant σ^2), then, since $\rho = 0$:

$$E^*(\widetilde{\eta}^K \mid (v_u)_{0 \le u \le T}, (S_u)_{0 \le u \le t}) = B(\widetilde{S}_t, \int_t^T v_u du)$$

and so the discounted price for payoff η^K is

$$\widetilde{C}_t^K = E^*(\widetilde{\eta}^K \mid \mathcal{F}_t) = E^*(B(\widetilde{S}_t, \int_t^T v_u du) \mid \mathcal{F}_t)$$
$$= f^K(\widetilde{S}_t, v_t, T - t)$$

which can be calculated if one knows a, b, c

HOW TO HEDGE IN THIS MODEL

Since the system is generated by two Brownian motions *under the risk neutral measure*: need two securities to hedge, say S_t, and C_t^X for one strike price X

If f_v^K means $f_v^K(\widetilde{S}_t, v_t, T-t)$, etc, and since $d[\widetilde{S}, v]_t = 0$

$$d\widetilde{C}_t^K = f_S^K d\widetilde{S}_t + \frac{1}{2} f_{SS}^K d[\widetilde{S}, \widetilde{S}]_t + f_v^K dv_t + \frac{1}{2} f_{vv}^K d[v, v]_t - f_t^K dt$$

and the same for C^X, so that

$$\begin{aligned} d\widetilde{C}_t^K &= f_S^K d\widetilde{S}_t + f_v^K dv_t + dt\text{-terms} \\ d\widetilde{C}_t^X &= f_S^X d\widetilde{S}_t + f_v^X dv_t + dt\text{-terms} \end{aligned}$$

Express the dv_t-term by

$$dv_t = \frac{1}{f_v^X} d\widetilde{C}_t^X - \frac{f_S^X}{f_v^X} d\widetilde{S}_t + dt\text{-terms}$$

and get

$$d\widetilde{C}_t^K = f_S^K d\widetilde{S}_t + f_v^K \left(\frac{1}{f_v^X} d\widetilde{C}_t^X - \frac{f_S^X}{f_v^X} d\widetilde{S}_t \right) + dt\text{-terms}$$

Rearrange:

$$d\widetilde{C}_t^K = \left(f_S^K - \frac{f_v^K f_S^X}{f_v^X} \right) d\widetilde{S}_t + \frac{f_v^K}{f_v^X} d\widetilde{C}_t^X + dt\text{-terms}$$

However, $\widetilde{C}_t^K$, $\widetilde{S}_t$ and $\widetilde{C}_t^X$ are all (local) martingales, so

$$dt\text{-terms} = 0$$

Final self financing strategy:

$$d\widetilde{C}_t^K = \left(f_S^K - \frac{f_v^K f_S^X}{f_v^X} \right) d\widetilde{S}_t + \frac{f_v^K}{f_v^X} d\widetilde{C}_t^X$$

in other words: to hedge payoff η^K, hold

- $\left(f_S^K - \frac{f_v^K f_S^X}{f_v^X} \right)$ units of stock S_t
- $\frac{f_v^K}{f_v^X}$ units of the option with payoff η^X

Initial starting capital: $f(\widetilde{S}_0, v_0, T)$ dollars

THE ONLY THING THAT REMAINS IS TO CALCULATE THE FUNCTION f

ANOTHER CASE OF HEDGING IN ADDITIONAL SECURITIES

The superreplication in HW 6

THE THREE INVARIANCES

SELF FINANCING STRATEGIES ARE INVARIANT UNDER:

- Change of numeraire
- Change of measure (so long as absolutely continuous)
- Change of time

CHANGE OF TIME

Function $f : [0, T] \to [0, T']$ is a time change if

- f is increasing
- $f(0) = 0$ and $f(T) = T'$
- $f^{(-1)}(t)$ is a stopping time, for each t

Securities on original time scale $\widetilde{S}_t$, $\widetilde{C}_t$ are connected by

$$\widetilde{C}_t = \widetilde{C}_0 + \int_0^t \theta_u d\widetilde{S}_u$$

Securities on new time scale

$$\widetilde{S}_t^{\text{new}} = \widetilde{S}_{f(t)} \quad \text{and} \quad \widetilde{C}_t^{\text{new}} = \widetilde{C}_{f(t)}$$

satisfy

$$\widetilde{C}_t^{\text{new}} = \widetilde{C}_0 + \int_0^t \theta_u^{\text{new}} d\widetilde{S}_u^{\text{new}}$$

where

$$\theta_t^{\text{new}} = \theta_{f(t)}$$

This is because the stochastic integral is a limit of sums:

Set $u_i = f(v_i)$:

$$\begin{aligned}
\widetilde{C}_t^{\text{new}} - \widetilde{C}_0 &= \int_0^{f(t)} \theta_u d\widetilde{S}_u \\
&\approx \sum_{u_i < f(t)} \theta_{u_i} (\widetilde{S}_{u_{i+1}} - \widetilde{S}_{u_i}) \\
&= \sum_{v_i < t} \theta_{f(v_i)} (\widetilde{S}_{f(v_{i+1})} - \widetilde{S}_{f(v_i)}) \\
&= \sum_{v_i < t} \theta_{v_i}^{\text{new}} (\widetilde{S}_{v_{i+1}}^{\text{new}} - S_{v_i}^{\text{new}}) \\
&\approx \int_0^t \theta_u^{\text{new}} d\widetilde{S}_u^{\text{new}}
\end{aligned}$$

IMPORTANT CONSEQUENCE OF INVARIANCE TO TIME CHANGE

Suppose that the discounted payoff $\widetilde{\eta}$ satisfies:

$$\widetilde{\eta} = g(\widetilde{S})$$

where g is invariant to time:

if $s_t^{\text{new}} = s_{f(t)}$ for any deterministic time change, then $g(s^{\text{new}}) = g(s)$

Example: European payoffs, barrier and lookback payoffs *written on the discounted process, or future*

THEN: if you know that $\int_0^T \sigma_t^2 dt = \Xi$, you can price option as if $\sigma_t^2 = \Xi/T = \text{constant}$

Technically: time change on the form:

$$f^{(-1)}(t) = \int_0^t \sigma_u^2 du$$

The time changed security is

$$d\widetilde{S}_t^{\text{new}} = \sqrt{\frac{\Xi}{T}} \widetilde{S}_t^{\text{new}} dW^{\text{new}}$$

(see p. 4-6 in Lect 5 (part 2))

CONVEXITY AND OPTIMAL STOPPING

JENSENS INEQUALITY

g is a convex function, M_t is a martingale, τ_1, τ_2 are stopping times, with $0 \le \tau_1 \le \tau_2 \le T$

$$E(g(M_{\tau_2}) \mid \mathcal{F}_{\tau_1}) \ge g(M_{\tau_1})$$

SEVERAL CONSEQUENCES, SUCH AS

APPLICATION 1: If the interest rate is zero, it is never optimal to exercise any convex payoff early. Application to superhedging in Lecture 5.

APPLICATION 2: The American option inequality (Lect 5 in Fall (p. 14))

Assume also $g(s) \ge 0$, $g(0) = 0$, and that $\exp\{-\int_0^t r_u du\}S_t$ is a martingale. Then

$$\exp\{-\int_0^t r_u du\}g(S_t) \text{ is a submartingale} \qquad .$$

In particular, it is never optimal to exercise an American call early (when there is no dividend)