HEDGING: THE EXACT CASE

SETUP: ON DISCOUNTED SCALE

\( \tilde{S}_t \): stock price \( \tilde{\eta} \): payoff at \( T \) \( \tilde{C}_t \): price at \( t \)

SELF FINANCING STRATEGY (SFS):

\[ \tilde{\eta} = \tilde{C}_T \]

\[ d\tilde{C}_t = \theta_t d\tilde{S}_t \quad \theta_t = \text{“delta”} \]

QUADRATIC VARIATION \([\cdot, \cdot]_t\):

\[ d[\tilde{C}, \tilde{S}]_t = \theta_t d[\tilde{S}, \tilde{S}]_t \]

OR:

\[ \theta_t = \frac{d[\tilde{C}, \tilde{S}]_t}{d[\tilde{S}, \tilde{S}]_t} \quad (\tilde{1}) \]

CONCLUSION: IF YOU KNOW HOW TO CALCULATE OPTIONS PRICES \( \tilde{C}_t \), AND IF SFS EXISTS, THEN YOU CAN CALCULATE \( \theta_t \).

WHAT DO YOU GET FROM (\( \tilde{1} \)) IF SFS DOES NOT EXIST?

NEXT PAGE ON ORIGINAL SCALE
NUMERAIRE INVARIANCE: SFS IS

\[ \eta = C_T \]

\[ dC_t = \theta_t^{(0)} d\Lambda_t + \theta_t^{(1)} dS_t \]  \hspace{1cm} (1)

\[ C_t = \theta_t^{(0)} \Lambda_t + \theta_t^{(1)} S_t \]  \hspace{1cm} (2)

IF \( \Lambda_t = \exp \left( \int_0^t r_u du \right) = \) MONEY MARKET BOND:

\( \Lambda_t \) IS \( dt \) TERM, SO

\[ dC_t = \theta_t^{(1)} dS_t + dt \) TERM

OR: \( \theta_t^{(1)} = \frac{d[C, S]_t}{d[S, S]_t} \)

\( \theta_t^{(0)} \) IS GIVEN BY (2)

SINCE

\[ C_t = e^{rt} \tilde{C}_t, \) OR \( dC_t = e^{rt} d\tilde{C}_t + dt \) TERM

\[ S_t = e^{rt} \tilde{S}_t, \) OR \( dS_t = e^{rt} d\tilde{S}_t + dt \) TERM

ONE OBTAINS

\[ d[C, S] = (e^{rt})^2 d[\tilde{C}, \tilde{S}] \hspace{1cm} d[S, S] = (e^{rt})^2 d[S, S]_t \]

WHENCE

\[ \theta_t^{(1)} = \frac{(e^{rt})^2 d[\tilde{C}, \tilde{S}]_t}{(e^{rt})^2 d[\tilde{S}, \tilde{S}]_t} = \theta_t \) FROM \( (\tilde{1}) \)

AS BEFORE
BLACK-SCHOLES MODEL:

\[ dS_t = rS_t dt + \sigma S_t dB_t \]

UNDER \( P^* \)

\[ \Lambda_t = \text{MONEY MARKET BOND, EUROPEAN OPTIONS} \]

\[ C_t = f(S_t, T - t) = \text{B-S formula} \]

\[ dC_t = f'_S(S_t, T - t) dS_t + dt \text{ terms} \]

so \( d[C, S]_t = f'_S(S_t, T - t) d[S, S]_t \)

OR \( \theta_t = \frac{d[C, S]_t}{d[S, S]_t} = f'_S(S_t, T - t) \)

= THE REGULAR B-S DELTA

SHOOTING SPARROWS WITH CANNON
AN ASIAN LIABILITY

\[ \eta = \int_0^T g(S_u)du \]

TAKE \( r = 0 \):

\[ C_t = E \left( \int_0^T g(S_u)du \mid \mathcal{F}_t \right) \]

\[ = E \left( \int_0^t g(S_u)du \mid \mathcal{F}_t \right) + E \left( \int_t^T g(S_u)du \mid \mathcal{F}_t \right) \]

\[ = \int_0^t g(S_u)du + \int_t^T f(S_t, u - t)du \]

WHERE \( f(s, t) = \) BS price at zero for payoff \( g(S_t) \) at \( t \).

SET \( h(s, t) = \int_0^t f(s, u)du \):

\[ C_t = \int_0^t g(S_u)du + h(S_t, T - t) \]

THEN \( dC_t = h'_S(S_t, T - t) + dt - \) TERMS

\[ \theta_t = \frac{d[C, S]}{d[S, S]_t} = h'_S(S_t, T - t) \]
MORE INTERESTING: A LOOKBACK OPTION

\[ \eta = (M_T - K)^+ \quad \text{WHERE} \quad M_t = \max_{0 \leq u \leq t} S_u \]

\[ C_t = e^{-r(T-t)} E^*(\eta \mid \mathcal{F}_t) = \text{PRICE AT } t \]

IN LECTURE 1, WE LEARNED TO CALCULATE \( C_0 \)
FOR FIXED \( r, \sigma^2 \). SET

\[ C_0 = f(s_0, K, T) \]

FIRST OBTAIN \( C_t \)

SET: \( K_t = \max(K, M_t) \)

NOTE: \( (M_T - K) = (M_T - K_t) + (K_t - K) \)

\( M_T \geq K_t \geq K \) WHEN \( M_T \geq K \):

\[ (M_T - K)^+ = (M_T - K_t)^+ + (K_t - K)^+ \]

SO

\[ C_t = e^{-r(T-t)} E^* \left[ (M_T - K_t)^+ \mid \mathcal{F}_t \right] \]
\[ + e^{-r(T-t)} E^+ \left[ (K_t - K)^+ \mid \mathcal{F}_t \right] \]
\[ = f(S_t, K_t, T - t) + e^{-r(T-t)}(K_t - K)^+ \]
LOOKBACK (CONTINUED)

\[ C_t = f(S_t, K_t, T - t) + e^{-r(T-t)}(K_t - K)^+ \]

FINDING THE HEDGE

\( K_t \) IS NONDECREASING (BUT NOT A \( dt \) TERM):

\[ dC_t = f'_S(S_t, K_t, T - t)dS_t \]

+ NONDECREASING (INCL \( dt \)) TERMS

SO: \( d[C, S]_t = f'_S(S_t, K_t, T - t)d[S, S]_t \)

FROM EITHER ABOVE EQUATION:

\[ \theta_t = f'_S(S_t, K_t, T - T) \]

THE DELTA IS STILL DERIVATIVE W.R.T. \( S \), BUT HARDER TO SEE DIRECTLY.

ALSO: \( K_t = \) FUNCTION OF PATH OF \( S_t \).
MORE COMPLEX LOOKBACKS

PAYOFF $g(M_T)$ at $T$

RECALL FROM LECTURE 1:

$$V_0 = f(s_0, T; g)$$

$$= e^{-rT} E[g(M_T)]$$

$$= e^{-rT} \int \int_{b \geq a} g(S_0 \exp(\sigma b))$$

$$\exp \left( \nu a - \frac{1}{2} \nu^2 T \right) f_{X,Mx}(a, b) dadb$$

where $\nu = \frac{1}{\sigma} \left( r - \frac{1}{2} \sigma^2 \right)$

and

$$f_{X,Mx}(a, b) = \begin{cases} 
2 \frac{(2b - a)}{\sqrt{2\pi T^3}} \exp \left( -\frac{(2b - a)^2}{2T} \right) & \text{if } b \geq a \\
0 & \text{otherwise}
\end{cases}$$
PRICE AT TIME $t$

Let $m = M_t$, $s = S_t$, $N = \max_{t \leq u \leq T} S_u$

$$M_T = \begin{cases} m & \text{on set } N < m \\ N & \text{on set } N \geq m \end{cases}$$

$$g(M_T) = g(m)I(N < m) + g(N)I(N \geq m)$$

$$E^*(g(M_T) \mid \mathcal{F}_t)$$

$$= g(m)P^*(N < m \mid S_t = s) + E^*[g(N)I(N \geq m) \mid S_t = s]$$

$$= g(m)P^*(M_{T-t} < m \mid S_0 = s) +$$

$$E^*[g(M_{T-t})I(M_{T-t} \geq m) \mid S_0 = s]$$

$$= e^{r(T-t)}[g(m)f(s, T-t; I(\cdot < m))$$

$$+ f(s, T-t, g(\cdot)I(\cdot \geq m))]$$
Set

\[ h(m, s, T) = g(m)f(s, T-t, I(\cdot < m)) + f(s, T-t, g(\cdot)I(\cdot \geq m)) \]

\[ = e^{-r(T-t)} \left\{ g(m) \int \int I(S_0 \exp(\sigma b) < m) \right. \]

\[ \exp \left( va - \frac{1}{2} v^2 T \right) f_{X,M}(a, b) dadb \]

\[ + \int \int g(S_0 \exp(\sigma b))I(S_0 \exp(\sigma b) \geq m) \]

\[ \left. \exp \left( va - \frac{1}{2} v^2 T \right) f_{X,M}(a, b) dadb \right\} \]

Then

\[ V_t = h(M_t, S_t, T - t) \]

\[ dV_t = h'_S dS_t + \text{terms with } d \text{ (increasing quantities)} \]

Again

\[ \text{delta} = f'_S(M_t, S_t, T - t) \quad (A) \]

**GENERAL PRINCIPLE**

If \( V_t = h(M_t, S_t, T - t) \) and \( M_t \) is any quantity without quadratic variation, then (A) holds

Most general form: Malliavin calculus
WHAT IF YOU DO NOT KNOW FORM OF $V_t$?

MONTE CARLO SIMULATION

FOR EXAMPLE, SUPPOSE

$(i)$ you know that $V_t = f(M_t, S_t, T - t)$ where $M_t =$ maximum, or any other quantity without quadratic variation

$(ii)$ you don’t know the form of $f$

$(iii)$ you know how to simulate $(V_T, M_t, S_t)$, but not $V_t$

PROCEDURE: SIMULATE $n$ COPIES $(V_T^{(i)}, M_t^{(i)}, S_t^{(i)})$

$i = 1, \ldots, n$ UNDER $P^*$

Since $f(m, s, T - t) = E^*(V_T|M_t = m, S_t = s)$, use estimate

$\hat{f}(m, s, T - t) = \text{NONPARAMETRIC REGRESSION}$

of $V_T^{(i)}$ on $M_t^{(i)}, S_t^{(i)}$
KERNEL REGRESSION
NADARAYA-WATSON ESTIMATOR

\[ K(m, s) = \text{"KERNEL"} \]
\[ = \text{any density in } m, s \]
\[ \hat{f}(m, s, T-t) = } \frac{\sum V_T^{(i)} K(M_t^{(i)} - m, S_t^{(i)} - s)}{\sum K(M_t^{(i)} - m, S_t^{(i)} - s)} \]
\[ \Delta = \hat{f}'_S(m, s, T-t) \]

FOR EXAMPLE,

\[ K(m, s) = \frac{1}{h_1 h_2} \phi \left( \frac{m}{h_1} \right) \phi \left( \frac{s}{h_2} \right) \]

FOR \( n \to \infty \), OPTIMAL \( h_1, h_2 \to 0 \)

- MUCH THEORY AVAILABLE
- OTHER ESTIMATORS AVAILABLE: Local linear regression, LOcally WEighted Scatter plot Smoothing (LOWESS)
- NEED TO REDUCE \# ARGUMENTS TO MINIMUM
HOW DO YOU KNOW YOU HAVE THE RIGHT ARGUMENTS?

\[ \widehat{f}'(M_t, S_t, T-t) = \text{CORRECT DELTA FOR PAYOFF } g(M_T) \]

\[ \widehat{f}'(S_t, T-t) = \text{WRONG DELTA FOR PAYOFF } g(M_T) \]

TO TEST THIS:

- OBTAIN \( \widehat{f}'(m, s, T-t) \) FOR SEVERAL \( t \) (REUSE SAMPLE FOR DIFFERENT \( t \), OR REDUCE DIMENSION)
- DO NEW SIMULATION TO SEE DISTRIBUTION OF

\[ g(M_T^{(new,i)}) - \sum_{t_j} \widehat{f}'(M_{t_j}^{(new,i)}, S_{t_j}^{(new,i)}, T-t_j) \Delta S_{t_j}^{(new,i)} \]

IF \( \approx 0 \): YOU’RE OK

\( \widehat{f}(s, T-t) \) FAILS TEST FOR \( g(M_T) \)
HEDGING A EUROPEAN OPTION PAYOFF $g(S_T)$ UNDER INCOMPLETENESS (ONE STRATEGY)

\[
C(s, R, \equiv) = \exp(-R)E_Qg(s \times \exp(R-\equiv/2 + \sqrt{\equiv}Z)
\]

(1)

Black-Scholes price $= C(S_t, r(T - t), \sigma^2(T - t))$

If $\equiv^+ \geq \int_0^T \sigma_u^2 du$

\[
d\tilde{C} \left( \tilde{S}_t, 0, \equiv^+ - \int_0^t \sigma_u^2 du \right)
\]

\[
= \tilde{C}'_S d\tilde{S}_t + \frac{1}{2} \tilde{C}''_{SS} d[S, S]_t - \tilde{C}_\equiv \sigma_t^2 dt
\]

\[
= \underbrace{\tilde{C}'_S d\tilde{S}_t}_{\text{HEDGEABLE}} + \left( \frac{1}{2} \tilde{C}''_{SS} \tilde{S}^2_t - \tilde{C}'_\equiv \right) \sigma_t^2 dt = 0
\]

BECAUSE, FOR ALL $s, \equiv$,

\[
C'_\equiv(s, 0, \equiv) = \frac{1}{2} C''_{SS}(s, 0, \equiv)s^2
\]

BY DIFFERENTIATION OF (1)