ADDENDUM TO LECTURE 15

1. Case Study on Föllmer and Leukert (1999)

Let's price a lookback option using the methods in this paper. We shall mainly be working with the European call example from Section 3. The payoff is

$$H = g(\max_{0\le t\le T} X_t), \tag{1}$$

where g is a nondecreasing function, such as $g(m) = (m-K)^+$.

From the development in Section 3, we know that $dP^*/dP = c_1 X_T^{-m/\sigma^2}$. Hence the set A can be written

$$A = \{c_1^{-1} X_T^{m/\sigma^2} > c_2 g(M_T)\} = \{X_T^{m/\sigma^2} > c_3 g(M_T)\}. \tag{2}$$

As in the paper, there will be two different cases depending on whether m/σ^2 is bigger or smaller than 1.

What we then need to do is to calculate P(A). For this purpose, one can use the known results about the joint distribution of a brownian motion and its running maximum. Suppose that Z_t is a Brownian motion and the maximum is given by

$$M_t \;=\; \max_{0\le u\le t} Z_u. \tag{3}$$

Then, the joint density of Z_T and M_T is

$$f_{Z,M}(z,b) = -\frac{2}{T}\phi'\left(\frac{(2b-z)}{\sqrt{T}}\right) \text{ for } b \ge z^+ \tag{4}$$

where ϕ is the standard normal density (and so $\phi'(x) = -x\phi(x)$). $f_{Z,M}(z,b) = 0$ if $b<0$ or $b<z$. See Karatzas and Shreve (1987), p. 95.

Of course, X is not a Brownian motion, but if we set $Z_t = \sigma^{-1}\log(X_t/X_0)$, then we can use Girsanov to force Z to become one. Since, under P

$$Z_t = \theta t + W_t \tag{5}$$

where $\theta = \frac{1}{\sigma}\left(m - \frac{\sigma^2}{2}\right)$, Z will be a Brownian motion under $\tilde{P}$ for which

$$\frac{dP}{d\tilde{P}} \;=\; \exp\{\theta Z_T - \frac{1}{2}\theta^2 T\}. \tag{6}$$

As usual,

$$\max_{0\le t\le T} X_t = X_0 \exp\{\sigma M_T\} \tag{7}$$

Finding A, therefore, boils down to calculating

$$P(A) = \tilde{E} I_A \frac{dP}{d\tilde{P}} \tag{8}$$

or

$$P(A) = \int\int_C \exp\{\theta z - \frac{1}{2}\theta^2 T\} f_{Z,M}(z,b)\,dz\,db \tag{9}$$

where C is the set so that $A = \{(Z, M) \in C\}$, *i.e.*,

$$C = \{(z, b) : \ \exp\{\frac{m}{\sigma^2}[\sigma z + \log X_0]\} \ > \ c_3 g(X_0 \exp\{\sigma b\})\}. \tag{10}$$

Delightful expression, but analytic expressions can be found for standard call style payoffs.

Once A has been determined from ϵ, then the price of the option becomes

$$E^*(HI_A) \ = \ \int\int_C g(X_0 \exp\{\sigma b\}) \exp\{\theta^* z - \frac{1}{2}\theta^{*2} T\} f_{Z,M}(z, b) dz db, \tag{11}$$

where θ^* is now such that

$$\frac{dP^*}{d\tilde{P}} \ = \ \exp\{\theta^* Z_T - \frac{1}{2}\theta^{*2} T\}. \tag{12}$$

2. References

Föllmer, H., and Leukert, P. (1999). Quantile hedging. *Finance and Stochastics* **3** 251-273.