
Let’s price a lookback option using the methods in this paper. We shall mainly be working with the European call example from Section 3. The payoff is

\[ H = g(\max_{0 \leq t \leq T} X_t), \]

where \( g \) is a nondecreasing function, such as \( g(m) = (m - K)^+ \).

From the development in Section 3, we know that \( dP^* / dP = c_1 X_T^{-m/\sigma^2} \). Hence the set \( A \) can be written

\[ A = \{ c_1^{-1} X_T^{m/\sigma^2} > c_2 g(M_T) \} = \{ X_T^{m/\sigma^2} > c_3 g(M_T) \}. \]

As in the paper, there will be two different cases depending on whether \( m/\sigma^2 \) is bigger or smaller than 1.

What we then need to do is to calculate \( P(A) \). For this purpose, one can use the known results about the joint distribution of a brownian motion and its running maximum. Suppose that \( Z_t \) is a Brownian motion and the maximum is given by

\[ M_t = \max_{0 \leq u \leq t} Z_u. \]

Then, the joint density of \( Z_T \) and \( M_T \) is

\[ f_{Z,M}(z,b) = -\frac{2}{T} \phi' \left( \frac{2b - z}{\sqrt{T}} \right) \text{ for } b \geq z^+ \]

where \( \phi \) is the standard normal density (and so \( \phi'(x) = -x\phi(x) \)). \( f_{Z,M}(z,b) = 0 \) if \( b < 0 \) or \( b < z \). See Karatzas and Shreve (1987), p. 95.

Of course, \( X \) is not a Brownian motion, but if we set \( Z_t = \sigma^{-1} \log(X_t/X_0) \), then we can use Girsanov to force \( Z \) to become one. Since, under \( P \)

\[ Z_t = \theta t + W_t \]

where \( \theta = \frac{1}{\sigma} \left( m - \frac{\sigma^2}{2} \right) \), \( Z \) will be a Brownian motion under \( \hat{P} \) for which

\[ \frac{dP}{d\hat{P}} = \exp\{\theta Z_T - \frac{1}{2} \theta^2 T\}. \]

As usual,

\[ \max_{0 \leq t \leq T} X_t = X_0 \exp\{\sigma M_T\} \]

Finding \( A \), therefore, boils down to calculating

\[ P(A) = \hat{E}_A \frac{dP}{d\hat{P}} \]

or

\[ P(A) = \int \int_C \exp\{\theta z - \frac{1}{2} \theta^2 T\} f_{Z,M}(z,b) dz db \]
where $C$ is the set so that $A = \{(Z, M) \in C\}$, i.e.,

$$C = \{(z, b) : \exp \left\{ \frac{m}{\sigma^2} \sigma z + \log X_0 \right\} > c_3 g(X_0 \exp \{\sigma b\}) \}.$$  

Delightful expression, but analytic expressions can be found for standard call style payoffs. Once $A$ has been determined from $\epsilon$, then the price of the option becomes

$$E^*(H I_A) = \int \int_C g(X_0 \exp \{\sigma b\}) \exp \{\theta^* z - \frac{1}{2} \theta^* \sigma^2 T\} f_{Z, M}(z, b) dz db,$$

where $\theta^*$ is now such that

$$\frac{dP^*}{dP} = \exp \{\theta^* Z_T - \frac{1}{2} \theta^* \sigma^2 T\}.$$  

2. References