MathFinance 346/Stat 391. Winter 2004.

Homework for Lecture 7. Due Fri 3/5 at the beginning of the review session.

1. **The digital band option.**

Consider a discounted stock price $\tilde{S}$ with unknown and time varying volatility σ_t^2. There are also no known bounds on this volatility. Consider strike prices K_1 and K_2, and we are interested in a discounted payoff at time T given by

$$\tilde{\eta} = \text{ amount of time } t \in [0,T] \text{ so that } K_1 \leq \tilde{S}_t \leq K_2.$$

(a) Given the existence of a market traded European put with (discounted) strike price K, $K_1 < K < K_2$, given an approximate hedge for $\tilde{\eta}$. Use a limiting argument to show the validity of the approximation.

(b) What whould happen if there are market traded puts with (discounted) strike price K, *for all* $K_1 < K < K_2$?

2. **The American down and out put.** Let the volatility σ and the short rate r be fixed. Consider the option η which pays $(K - S_\tau)^+$ if $\min_{t \leq \tau} S_t > H$, and 0 otherwise. The exercise time τ is, as usual, determined by the option owner, and must satisfy $0 \leq \tau \leq T$.

(a) State a formula for the price of η at time $t < T$.

(b) Describe, as well as you can, the shape of $\mathcal{C}$, $\mathcal{S}$, and S_t^* (the continuation and stopping regions, and the exercise boundary).

3. **The essential supremum.**

On a probability space $(\Omega, \mathcal{F}, P)$, let $\{X_\alpha\}$ be a collection of random variables.

We define X to be "ess $\sup_\alpha X_\alpha$" if (I) $X \geq X_\alpha$ a.s. for all α, and provided (II) any Y that satisfies $Y \geq X_\alpha$ a.s. for all α also satisfies $Y \geq X$ a.s.

(a) Show that if the collection $\{X_\alpha\}$ is countable, i.e., on the form $\{X_n, n = 1, 2, ...\}$, then ess $\sup_n X_n = \lim_{n\to\infty} \max(X_1, ..., X_n)$, almost surely.

(b) In the following let $\mathcal{P}^*$ be the set of all risk neutral measures equivalent to P. Suppose that there is a countable subset $\mathcal{Q}^* \subset \mathcal{P}^*$ so that for every $P^* \in \mathcal{P}^*$, there is a sequence Q_n^* in $\mathcal{Q}^*$ so that dQ_n^*/dP converges a.s. to dP^*/dP. Also let $\mathcal{G} \subset \mathcal{F}$, and let Z be bounded. Show that, in this case, $E_{Q_n^*}(Z|\mathcal{G}) \to E_{P^*}(Z|\mathcal{G})$.

(c) In the setting of (b), find and prove a result similar to (a) to approximate ess $\sup_{P^* \in \mathcal{P}^*} E_{P^*}(Z|\mathcal{G})$ with a countable sequence.